# ON THE ARITHMETIC OF TILED ORDERS

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# Abstract

This thesis investigates arithmetic and algebraic properties of tiled orders in central simple algebras over non-archimedean local fields. To this end, we make extensive use of a building-theoretic framework that allows us to gain combinatorial and geometric intuition for properties of these local orders. We use this local information and turn to a global setting, where we compute type numbers of global orders. We accomplish this with tools from algebraic number theory, as well as class field theory towards the end of the thesis.

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# Contents

# Chapter 1

# Introduction

In this thesis, we embark on an exploration of arithmetic and algebraic properties of tiled orders, which are a class of orders in central simple algebras. During this journey, we will travel back and forth between two different realms: one of them is algebraic, in which we eventually want our results to reside, and the other is rather combinatorial and geometric, from which we will often borrow intuition.

Tiled orders are known under various names and have been studied in many contexts. In particular, we borrow the name *tiled* as initially coined by Tarsy in [41]. As tiled orders, Fujita [11], Jategaonkar [21], Rump [38], Tarsy [41] and others have studied their global dimensions. Tiled orders are also called *graduated* orders in the work of Plesken [33] on representation theory of finite groups. The explicit construction in this thesis is based on the work of Shemanske [39], where such orders are called *split*, as also seen in Hijikata [19].

The setting in which we study tiled orders are central simple algebras $M_n(D)$, where $D$ is a division algebra with center $k$ a non-archimedean local field. While we will give a precise definition of tiled orders in Chapter 3, they encompass among other classes maximal and hereditary orders, and are therefore quite general. In particular, tiled orders can be written as an intersection of finitely many maximal orders in $M_n(D)$. In [39], Shemanske associates to a given tiled order $\Gamma$ a convex polytope $C_\Gamma$ in an apartment $\mathcal{A}$ of the building for $SL_n(D)$. As shown in [40], $C_\Gamma$ is uniquely determined by a nice set of vertices, which on the one hand correspond to some maximal orders containing $\Gamma$, and on the other hand to a set of $\Gamma$-lattices as introduced by Plesken in [33]. We will call such vertices distinguished, and they will provide the first door between the algebraic and the geometric contexts.

In [40], it is shown that the normalizer $\mathcal{N}(\Gamma)$ of $\Gamma$ in $GL_n(D)$ induces actions on the polytope $C_\Gamma$, and these actions are given by a subgroup of a certain symmetric group. We extend this result in Corollary 3.2.10, where we view isomorphisms of tiled orders in terms of actions on the apartment $\mathcal{A}$ by sending one associated polytope to the other. In particular, we can study algebraic properties of any given isomorphism class of tiled orders by studying rigid motions on the associated polytope. We refer to these actions on a given polytope as *symmetries of* $C_\Gamma$.

At the heart of accomplishing this goal lies a family of easily computable non-negative integers called *structural invariants*. They were introduced by Zassenhaus in [47], who has shown that they determine the isomorphism class of $\Gamma$. We prove that the structural invariants also encode geometric data for $C_\Gamma$, by giving distances between distinguished vertices and certain hyperplanes bounding $C_\Gamma$. Therefore, the "shape" and "size" of $C_\Gamma$ is encoded in these structural invariants. In particular, two tiled orders are isomorphic if and only if their polytopes are congruent, where congruence is defined in terms of structural invariants. We take this connection between isomorphisms of tiled orders and structural invariants further in Proposition 3.7.8, which is a stepping stone for most of our results. Specifically, we extract information about the normalizer $\mathcal{N}(\Gamma)$ from the isomorphism class of $\Gamma$ :

**Proposition.** *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a tiled order, and $\{m_{ij\ell} \,|\, i, j, \ell \leq n\}$ its set of structural invariants. Then $\mathcal{N}(\Gamma) = \bigcup_{\sigma \in H} \xi_\sigma D^\times \Gamma^\times$, where $H$ is the subgroup of the symmetric group $S_n$ given by $H = \{\sigma \in S_n \,|\, m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq n\}$, and $\xi_\sigma = (\boldsymbol{\pi}^{\mu_{i1} - \mu_{\sigma(i)\sigma(1)}} \delta_{\sigma(i)j})$ for $\delta_{ij}$ the Kronecker delta.*

*In addition, if $\Gamma$ has full geometric rank, let $\phi : \mathcal{N}(\Gamma) \to S_n$ be the homomorphism defined in Proposition 3.7.4. Then $\mathcal{N}(\Gamma) = \bigsqcup_{\sigma \in H} \xi_\sigma D^\times \Gamma^\times$, and $\phi(\xi_\sigma) = \sigma$.*

Now that we have such a powerful tool to study the polytopes $C_\Gamma$, we stop along the way on a few previously treaded paths and topics of interest for the study of tiled orders. Harada has shown in [16] and [17] that if an order in $M_n(D)$ is hereditary, then it is tiled; we show in Proposition 3.4.3 that an order $\Gamma$ is hereditary if and only if its polytope is a simplex in the associated building. Of particular interest are hereditary orders whose polytopes are maximal simplices in the building called *chambers*. In this case, we show in Corollary 3.9.7 that $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong \mathbb{Z}/n\mathbb{Z}$.

Next, we turn to radical idealizer chains of orders, which are canonical chains of overorders terminating in hereditary orders. Proposition 3.5.6 gives a geometric interpretation for each of the overorders in the chain. A possible application of this result is getting bounds on the length of the radical idealizer chain for a tiled order, as has been done for other classes of orders by Nebe in [28].

Another interesting tool for the study of tiled orders are associated quivers, defined by Roggenkamp and Wiedemann in [46] and by Müller in [27]. There are two constructions that will be of interest: the unvalued quiver, denoted by $Q(\Gamma)$, and the valued quiver, denoted by $Q_v(\Gamma)$. One may use the structural invariants to find the quivers, so it is not surprising that we may extract some information regarding isomorphism classes of tiled orders from the associated quivers. In [15], Haefner and Pappacena prove that certain automorphisms of $\Gamma$ give automorphisms of $Q(\Gamma)$, but that they do not extend to automorphisms of the valued quiver $Q_v(\Gamma)$.

To solve this problem, we connect many of these tools for the study of tiled orders with an interesting construction, detailed in Theorem 3.9.3:

**Theorem.** *Given a tiled order* $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ *with structural invariants* $\{m_{ij\ell}\}$*, define* $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ *where* $\nu_{ij} = \sum_{\ell=1}^{n} m_{ij\ell}$*. Then* $\Gamma_c$ *is a centered tiled order with structural invariants* $\widetilde{m}_{ij\ell} = n \cdot m_{ij\ell}$ *for all* $1 \leq i, j, \ell \leq n$*, and* $\sigma \in S_n$ *is a symmetry of* $C_\Gamma$ *if and only if* $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$*.*

Geometrically, the associated tiled order $\Gamma_c$ is obtained by scaling the polytope $C_\Gamma$ and then translating it in a convenient manner, as described in Corollary 3.9.5. Therefore, its polytope has the same symmetries as $C_\Gamma$. As shown in Theorem 3.9.8, these symmetries also induce automorphisms of the valued quiver $Q_v(\Gamma_c)$.

**Theorem.** *Given a centered tiled order* $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ *in* $M_n(k)$*, there is a bijection between* $\mathrm{Aut}(Q_v(\Gamma_c))$ *and the symmetries of* $C_{\Gamma_c}$*.*

It is via this construction that we conclude that not all subgroups of $S_n$ are realizable as symmetry groups of convex polytoes $C_\Gamma$, a particular case being two-transitive proper subgroups of $S_n$. However, there is more to study; once we have $n \geq 5$, there are subgroups of $S_n$ that are not two-transitive (or direct extensions of such subgroups), and that cannot be symmetry groups of convex polytopes $C_\Gamma$.

Another class of tiled orders that has been of some interest are Gorenstein tiled orders, which sometimes come equipped with a permutation in $S_n$ called a Kirichenko permutation. As a consequence of Proposition 3.7.8, these permutations induce an automorphism of the orders and therefore an automorphism of the associated quiver. In fact, the quivers of Gorenstein tiled orders have relatively few arrows and nice automorphisms, which make them an interesting subject of study, as seen in papers such as [35], [5] and [6]. It would be interesting to study further how the structural invariants and the associated polytopes might inform properties of Gorenstein tiled orders.

Next, we employ a global setting. Armed with knowledge about normalizers of tiled orders, we can turn to studying global orders in central simple algebras over number fields. In particular, the orders we study are tiled at each finite place. By the Skolem-Noether theorem, two orders $\Lambda$ and $\Lambda'$ are (everywhere) locally isomorphic if and only if they are locally everywhere conjugate, in which case we say they are in the same *genus*. Since these local isomorphisms do not necessarily lift to a global isomorphism, a natural question to consider is determining the number of global isomorphism classes inside the genus of $\Lambda$. We call this number the *type number* of $\Lambda$, and will denote it by $G(\Lambda)$.

Type numbers of orders in central simple algebras have been investigated in a a few different contexts. Deuring [7] originally investigated type numbers of maximal orders in quaternion algebras with some specific discriminants. Type numbers of Eichler orders in definite quaternion algebras have been studied by Eichler [8], Peters [30] and Pizer [31], [32]. Furthermore, Vignéras considers Eichler orders in not totally definite quaternion algebras in [44].

The case for global algebras $A$ of degree $n \geq 3$ over a number field $K$ has been considered in [25], which investigates the genus of maximal orders. Since the degree of $A$ over $K$ is at least 3, $A$ satisfies the Eichler condition. Therefore, one can apply strong approximation and express the arithmetic of the global order in terms of idelic arithmetic over the field $K$. Consider an order $\Lambda$ in $A$ that is (everywhere) locally tiled. Denote the $A$- and $K$-idele groups by $J_A$ and $J_K$, let $\Omega$ be the set of infinite places ramifying in $A$ so $K_\Omega = \{a \in K : \nu(a) > 0 \text{ for all } \nu \in \Omega\}$, the normalizer of $\Lambda_\nu$ by $\mathcal{N}(\Lambda_\nu)$, and the restricted product $\prod'_\nu \mathcal{N}(\Lambda_\nu) := J_A \cap \prod_\nu \mathcal{N}(\Lambda_\nu)$. Then the type number $G(\Lambda)$ is given by the number of double cosets $A^\times \backslash J_A / \prod'_\nu \mathcal{N}(\Lambda_\nu)$. As a consequence of Eichler's theorem/strong approximation, the reduced norm induces a bijection

$$\mathrm{nr} : A^\times \backslash J_A / \prod_\nu{}' \mathcal{N}(\Lambda_\nu) \to K_\Omega^\times \backslash J_K / \mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Lambda_\nu)) \cong J_K / K_\Omega^\times \, \mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Lambda_\nu)).$$

For more details about Eichler's theorem and the notation above, see [34, Chapter 34] and [45, Section 28.4]. By Theorem 3.1 in [25], the codomain of the nr map above is a finite abelian group of exponent $n$.

In order to find the number of idelic cosets above, we need to find $\mathrm{nr}(\mathcal{N}(\Lambda_\nu))$ at each finite place. Proposition 4.2.2 gives an algebraic expression for $\mathrm{nr}(\mathcal{N}(\Lambda_\nu))$, which requires knowledge for the normalizer $\mathcal{N}(\Lambda_\nu)$ as described in Proposition 3.7.8. However, computing the normalizer is rather difficult, as we will see in Chapter 3, so instead we use a geometric approach to finding $\mathrm{nr}(\mathcal{N}(\Lambda_\nu))$. Since isomorphic tiled orders have congruent polytopes, one can partition the convex polytopes congruent to $C_{\Lambda_\nu}$ into "reflection classes", where two polytopes are in the same reflection class if there exists a product of reflections sending one polytope to the other. Theorem 4.3.7 is one of the main results of the thesis, where we show find $\mathrm{nr}(\mathcal{N}(\Lambda_\nu))$ in terms of reflection classes. In particular, for a tiled order $\Gamma$ in $M_n(D)$ where $D$ is a division algebra with uniformizer $\boldsymbol{\pi} \in D$ over a non-archimedean local field $k$ with valuation ring $R$, we have

**Theorem.** *Let $\Gamma$ be a tiled order with tuple $(m_{ij\ell})$ of structural invariants in lexicographical order, and ordered tuple of types of distinguished vertices $(t_1, t_2, \ldots, t_n)$. Let $\xi_s := diag(\boldsymbol{\pi}^s, 1, \ldots, 1)$ and $\Gamma_s := \xi_s \Gamma \xi_s^{-1}$. Then there are at most $n$ reflection classes of polytopes congruent to $C_\Gamma$, corresponding to the classes of orders*

$$\begin{aligned}
[\Gamma] = [\Gamma_0] &= [(m_{ij\ell}), (t_1, t_2, \ldots, t_n)] \\
[\Gamma_1] &= [(m_{ij\ell}), (t_1 + 1, t_2 + 1, \ldots, t_n + 1)] \\
[\Gamma_2] &= [(m_{ij\ell}), (t_1 + 2, t_2 + 2, \ldots, t_n + 2)] \\
&\vdots \\
[\Gamma_{n-1}] &= [(m_{ij\ell}), (t_1 + n - 1, t_2 + n - 1, \ldots, t_n + n - 1)].
\end{aligned}$$

*In particular, $\mathrm{nr}(\mathcal{N}(\Gamma)) = (k^\times)^d R^\times$ if and only if there are $d$ distinct reflection classes and $[\Gamma_s] = [\Gamma_t]$ for $s \equiv t \pmod d$.*

We get a full process for finding type numbers in algebras of prime degree $p$ over $K$. In particular, at each finite place, there are either one or $p$ such reflection classes, as shown in Corollary 4.3.9. To determine what is the case at each place, we may employ the algorithm in Remark 4.3.11, which allows us to avoid the issue of finding the full group of symmetries of the associated polytope. Since any order is maximal at all but finitely many places, we only need to repeat this process finitely many times. Once we have the number of the reflection classes at each prime, we can use Theorem 4.4.5 to find the type number of the original global order.

The general approach for finding type numbers starts with a general algorithm to find the number of reflection classes as outlined in Remark 4.3.14. Unfortunately, this algorithm doesn't completely circumvent us from looking for the normalizer $\mathcal{N}(\Lambda_\nu)$. Having this information in hand, we obtain a general class number formula for finding type numbers in Theorem 4.4.11.

Many of the results in this thesis can be generalized to algebras over general global fields, however there are various cases that require caution. In addition, not all such algebras satisfy the Eichler condition, in which case other sets of tools would be necessary for finding type numbers. Some steps towards a generalization can be found in Brzezinski [4]. Finally, we could also look at $\mathcal{O}$-orders in $A$ where $\mathcal{O}$ is an arbitrary order in the number field $K$, but we would need to be careful with their associated class groups.

# Chapter 2

# Preliminaries

In this chapter, we introduce background concepts and establish notation to be used throughout. We start with some background in algebraic number theory, for which we will mostly follow the conventions in [29]. For the idelic theoretic notions, we follow [24]. Next, we introduce orders in central simple algebras, for which [34] will be the standard reference. Finally, the bulk of the chapter will consist of an introduction to a specific class of buildings that will be of interest to us, and we will make use of results stated in [1], [14, Chapter 19], and [36, Chapter 9].

## Section 2.1

## Algebraic number theory

Let $K$ be a number field with ring of integers $\mathcal{O}_K$. A **place** (or **prime**) of $K$ is an equivalence class of valuations on $K$; the set of places of $K$ is denoted by $\mathrm{Pl}(K)$. We will use the terms "place" and "prime" interchangeably. The archimedean places are also called **infinite** places, and the nonarchimedean places are also referred to as **finite** places. Denote the completions of $K$ and $\mathcal{O}_K$ with respect to a place $\nu \in \mathrm{Pl}(K)$

by $K_\nu$ and respectively $\mathcal{O}_\nu$. If $\nu$ is an infinite place, we set $\mathcal{O}_\nu := K_\nu$. At each finite place, $K_\nu$ is a local field with $\mathcal{O}_\nu$ its valuation ring. Let $\pi_\nu$ be a uniformizer of $\mathcal{O}_\nu$.

Next, we introduce some idelic language. Given a finite set of places $S$ of $K$, we define the set of **$S$-ideles** by

$$J_{K,S} := \prod_{\nu \in S} K_\nu^\times \prod_{\nu \notin S} \mathcal{O}_\nu^\times,$$

where the superscript $^\times$ denotes the (multiplicative) unit group.

We denote the **idele group of $K$** by

$$J_K := \bigcup_{\substack{S \subseteq \mathrm{Pl}(K) \\ S \text{ finite}}} J_{K,S} \subseteq \prod_\nu K_\nu^\times.$$

Then

$$J_K = \{\alpha = (a_\nu)_\nu \in \prod_\nu K_\nu^\times : a_\nu \in \mathcal{O}_\nu^\times \text{ for all but finitely many } \nu\} =: \prod_\nu{}' K_\nu^\times,$$

where $\prod_\nu'$ is the *restricted product* over the places $\nu$ of $K$.

### 2.1.1. Class groups

A **fractional ideal** of $K$ is a nonzero, finitely generated $\mathcal{O}_K$-submodule $\mathfrak{a} \subseteq K$. The set of fractional ideals forms an abelian group $I_K$. The identity element is $(1) = \mathcal{O}_K$, and inverses are given by

$$\mathfrak{a}^{-1} = \{x \in K \mid x\mathfrak{a} \subseteq \mathcal{O}_K\}.$$

Every fractional ideal admits a unique factorization into powers of prime ideals

$$\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}}$$

with $v_{\mathfrak{p}} \in \mathbb{Z}$ and $v_{\mathfrak{p}} = 0$ for all but finitely many $\mathfrak{p}$. The fractional principal ideals $(a) = a\mathcal{O}_K, a \in K^\times$ form a subgroup of $I_K$, denoted by $P_K$. The quotient group

$$Cl(K) = I_K/P_K$$

is called the **class group** of $K$. The class group is finite when $K$ (see V.1 in [24]), and we call

$$h(K) := \#\,\mathrm{Cl}(K)$$

the **class number** of $K$.

We can express the class group idelically. Let $S_\infty$ be the set of infinite places of $K$, and $J_{K,S_\infty} = \prod_{\nu \in S_\infty} K_\nu^\times \prod_{\nu \notin S_\infty} \mathcal{O}_\nu^\times$ the $S_\infty$-ideles defined before. Then

$$\mathrm{Cl}(K) \cong J_K/K^\times J_{K,S_\infty}.$$

Similarly, given any finite set of places $T \supseteq S_\infty$ containing the infinite places of $K$, we may define the $T$-ideal class group $\mathrm{Cl}_T(K)$ by

$$\mathrm{Cl}_T(K) := J_K/K^\times J_{K,T}.$$

Note that $J_{K,T} = J_{K,S_\infty} \prod_{\nu \in T \setminus S_\infty} K_\nu^\times$, so $J_{K,S_\infty} \subseteq J_{K,T}$. It follows that there exists a sujective homomorphism $\mathrm{Cl}(K) \twoheadrightarrow \mathrm{Cl}_T(K)$, so taking the quotient by the kernel gives

$$\mathrm{Cl}_T(K) \cong \mathrm{Cl}(K)/\langle [\mathfrak{p}] : \mathfrak{p} \in T \setminus S_\infty \rangle.$$

**Example 2.1.1.** Let $K = \mathbb{Q}(a)$ where $a$ is a root of $f(x) = x^3 - 7$. Using SageMath [43], we see that $\mathrm{Cl}(K) \cong \mathbb{Z}/3\mathbb{Z}$. Now let $\mathfrak{p} = (5, a+2)$, which is a prime ideal and therefore corresponds to a place of $K$, and let $T := \{\mathfrak{p}\} \cup S_\infty$. Since $\mathfrak{p}$ is non-principal, all ideal classes in $\mathrm{Cl}(K)$ are generated by $\mathfrak{p}$, and therefore $\mathrm{Cl}_T(K)$ is trivial.

*Remark* 2.1.2. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals in $\mathcal{O}_K$ such that their ideal classes coincide, so $[\mathfrak{p}] = [\mathfrak{q}]$. Idelically, if we associate to $\mathfrak{p}$ the idele $(\dots, 1, \pi_\mathfrak{p}, 1, \dots)$, and to $\mathfrak{q}$ the idele $(\dots, 1, \pi_\mathfrak{q}, 1, \dots)$, where $\pi_\mathfrak{p} \in K_\mathfrak{p}$ and $\pi_\mathfrak{q} \in K_\mathfrak{q}$ are uniformizers, then $(\dots, 1, \pi_\mathfrak{p}, 1, \dots)K^\times J_{K,S_\infty} = (\dots, 1, \pi_\mathfrak{q}, 1, \dots)K^\times J_{K,S_\infty}$. But then for any unit $u \in \mathcal{O}_\mathfrak{p}^\times$ and integer $a \in \mathbb{Z}$, we have

$$(\dots, 1, \pi_\mathfrak{p}^a u, 1, \dots)K^\times J_{K,S_\infty} = (\dots, 1, \pi_\mathfrak{q}^a, 1, \dots)K^\times J_{K,S_\infty},$$

so particular $K_\mathfrak{p}^\times K^\times J_{K,S_\infty} \subseteq K_\mathfrak{q}^\times K^\times J_{K,S_\infty}$. Similarly, we prove the reverse containment. Let $T_1 = \{\mathfrak{p}\} \cup S_\infty$ and $T_2 = \{\mathfrak{q}\} \cup S_\infty$. Then $K^\times J_{K,T_1} = K^\times J_{K,T_2}$, and therefore $\mathrm{Cl}_{T_1}(K) = \mathrm{Cl}_{T_2}(K)$.

The final type of class groups we consider are **ray class groups**. Let $\Omega$ be a subset of real places of $K$, and let $K_\Omega := \{a \in K : \nu(a) > 0 \text{ for all } \nu \in \Omega\}$ be the subset of $K$ consisting of elements of $K$ that are positive at all places in $\Omega$. Let $P_\Omega = \{(a) : a \in K_\Omega^\times\}$ be the set of principal ideals generated by elements of $K_\Omega$. Define the ray class group modulo $\Omega$ to be the quotient $\mathrm{Cl}_\Omega(K) := I_K/P_\Omega$. Idelically (see [29, Prop.1.9, page 365])

$$\mathrm{Cl}_\Omega(K) \cong J_K / \left( K^\times \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu \in S_\infty - \Omega} K_\nu^\times \prod_{\nu\nmid\infty} \mathcal{O}_\nu \right).$$

For a finite set $S$ of places of $K$ such that $S_\infty \subseteq S \cup \Omega$, define

$$J_{K,S,\Omega} := \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu\notin S\cup\Omega} \mathcal{O}_\nu^\times.$$

Note that in our previous notation, $J_{K,S_\infty} = J_{K,S_\infty,\emptyset}$, so we drop the subscript $\Omega$ when $\Omega$ is empty. Similarly, we let

$$\mathrm{Cl}_{S,\Omega}(K) := J_K/K^\times J_{K,S,\Omega} \cong \mathrm{Cl}_\Omega(K)/\langle[\mathfrak{p}] : \mathfrak{p} \in S\rangle.$$

Section 2.2

## Central simple algebras

In this section, we follow Sections 7 and 9 in [34]. Let $K$ be a field. Let $A$ be an algebra over $K$, and denote the center of $A$ by $A^c$.

**Definition 2.2.1.** A **central simple $K$-algebra** is a simple $K$-algebra $A$ for which $A^c = K$.

We call $D$ a **central division algebra** over $K$ if $D$ is a central simple algebra over $K$ such that and $D$ has no zero divisors.

Throughout, we suppose $A$ is finite dimensional over $K$, and we denote the **degree** of $A$ over $K$ by $\deg(A) := \sqrt{\dim_K A}$.

Any element $a \in A^\times$ determines an inner automorphism of $A$, where $x \mapsto axa^{-1}$.

The Skolem-Noether Theorem asserts that the converse is true, as well:

**Theorem 2.2.2** (Skolem–Noether, see (7.21) in [34]). *Let $K \subset B \subset A$, where $B$ is a simple subring of a central simple algebra $A$. Then every $K$-isomorphism $\phi$ of $B$ onto a subalgebra $\widetilde{B}$ of $A$ extends to an inner automorphism of $A$, that is, there exists an element $a \in A^{\times}$ such that*

$$\phi(b) = aba^{-1}, \qquad \textit{for all } b \in B.$$

An important corollary is as follows.

**Corollary 2.2.3** ((7.23), [34]). *Any $K$-automorphism of $A$ is inner.*

The Artin-Wedderburn theorem allows us to classify central simple algebras over $K$, and states that every central simple $K$-algebra is isomorphic to $M_n(D)$ for a unique central division algebra $D$ over $K$ up to $K$-algebra isomorphism. When $D$ is of degree $m$ over $K$, the degree of $A$ over $K$ is $\deg(A) = n\deg(D) = mn$.

Let $a \in A$, and consider the characteristic polynomial of the linear map corresponding to left multiplication by $a$ on $A$,

$$\text{charpoly}_{A/K} a = X^{m^2n^2} - (T_{A/K} a) X^{m^2n^2-1} + \cdots + (-1)^{m^2n^2} N_{A/K} a,$$

where $T_{A/K}$ is the trace map, and $N_{A/K}$ is the norm map.

In the case of central simple algebras, we also have *reduced trace* and *reduced norm* maps, defined as follows. Under the previous conditions, by Theorems (7.15) and (7.18) in [34], there exists a separable extension $E$ of $K$, called a *splitting field* of $A$, such that there is an isomorphism of $E$-algebras $h : E \otimes_K A \cong M_{mn}(E)$. Given an element $a \in A$, define the *reduced characteristic polynomial* of $a$ as the characteristic polynomial of the matrix $h(1 \otimes a)$,

$$\text{red.charpoly}_{A/K} a := \text{charpoly}\, h(1 \otimes a).$$

By Theorem (9.5) in [34], we get $\text{charpoly}_{A/K} a = (\text{red.charpoly}_{A/K} a)^{mn}$. Then the reduced characteristic polynomial is given by

$$\text{red.charpoly}_{A/K} a = X^{mn} - \text{tr}_{A/K}(a) X^{mn-1} + \cdots + (-1)^{mn}\, \text{nr}_{A/K}(a),$$

where

$$T_{A/K} a = mn \cdot \text{tr}_{A/K}(a), \qquad N_{A/K} a = (\text{nr}_{A/K}(a))^{mn}.$$

We call $\text{nr}_{A/K}(a)$ the *reduced norm* of $a$, and it does not depend on the choice of splitting field $E$ or isomorphism $h$. In particular,

$$\text{nr}_{A/K}(a) = \det(h(1 \otimes a))$$

is given by the determinant of the matrix in $h(1 \otimes a) \in M_{mn}(E)$, and is therefore multiplicative.

### 2.2.1. Central simple algebras over local fields

The material from this particular subsection is from Sections 12-14 of [34]. Let $A$ be a central simple algebra over a non-archimedean local field $k$ (with $\text{char}\, k = 0$) with a valuation $v$ and valuation ring $R$, unique maximal ideal $P$ and uniformizer $\pi$, such that the residue field $\overline{R} := R/P$ is finite. By Artin-Wedderburn, $A \cong M_n(D)$ for a central division algebra $D$ over $k$ of some degree $m$, so $\deg(A) = mn$. Let $\Delta$ be the unique maximal $R$-order in $D$, equipped with a prime element $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}^m = \pi$.

We have a normalized valuation $v_D$ on $D$, such that (see Equation (13.1) on page 139 of [34])

$$v_D(a) = \frac{1}{m} v(N_{D/k} a) = \frac{1}{m} v((\operatorname{nr}_{D/k} a)^m) = v(\operatorname{nr}_{D/k} a). \tag{2.1}$$

As discussed in [34, Section 14], $D$ (and $\Delta$) contain a maximal subfield $W$, which is an unramified extension of $k$. Moreover, $W$ is a splitting field for $D$, so $D \otimes_k W \cong M_m(W)$. In particular, there is an embedding $\mu : D \hookrightarrow M_m(W)$ which maps $\boldsymbol{\pi}$ to

$$\boldsymbol{\pi} \mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & 1 \\ \pi & 0 & 0 & \dots & 0 \end{pmatrix},$$

so $\operatorname{nr}_{D/k}(\boldsymbol{\pi}) = (-1)^{m-1}\pi$. In addition, this embedding gives $\operatorname{nr}_{D/k}(\alpha) = N_{W/k}(\alpha)$ for all $\alpha \in W$. Since $N_{W/k}$ maps the units of the valuation ring $R_W$ onto $R^\times$ (see Corollary (1.2) on page 319 in [29]), and $W$ is contained in $\Delta$, we have $\operatorname{nr}(\Delta) = R$.

The map $\mu$ induces an embedding $\widehat{\mu} : M_n(D) \hookrightarrow M_{mn}(W)$ where $(a_{ij}) \in M_n(D)$ and $(a_{ij}) \mapsto (\mu(a_{ij}))$. By definition, the reduced norm $\operatorname{nr}_{M_n(D)/k}(x) = \det(\widehat{\mu}(x))$. In particular, by embedding $D$ on the diagonal in $M_n(D)$, it follows that

$$\operatorname{nr}_{M_n(D)/k}(x) = (\operatorname{nr}_{D/k}(x))^n \qquad \text{for all} \quad x \in D \tag{2.2}$$

Another map on matrices over division rings is the Dieudonné determinant:

**Theorem 2.2.4** (Theorem 2.2.5 [37]). *There exists a unique "determinant map"*

$$\det : GL_n(D) \to D^\times/[D^\times, D^\times],$$

*where* $[D^\times, D^\times]$ *is the commutator subgroup of* $D^\times$. *The Dieudonné determinant has the following properties:*

*(a) The determinant is invariant under row additions.*

*(b) The determinant of the identity matrix is* 1.

*(c) If* $A \in GL_n(D)$, $a \in D^\times$, *and* $A'$ *is obtained from* $A$ *by (left-)multiplying one of the rows of* $A$ *by* $a$, *then*

$$\det(A') = \bar{a} \det A,$$

*where* $\bar{a}$ *denotes the image of* $a$ *in* $D^\times/[D^\times, D^\times]$.

*(d) If* $A, B \in GL_n(D)$, *then*

$$\det(AB) = \det(A)\det(B).$$

*(e) If* $A \in GL_n(D)$, *and if* $A'$ *is obtained from* $A$ *by interchanging two of its rows, then*

$$\det A' = -\det A.$$

**Definition 2.2.5.** Define $SL_n(D) := \ker \det$, the kernel of the determinant map.

We finish this subsection with the following connection between reduced norms and the Dieudonné determinant in the algebra $GL_n(D)$.

**Lemma 2.2.6.** *Let $v$ and $v_D$ be (normalized) valuations on $k$, and $D$, and let $x \in GL_n(D)$. Then*

$$v_D(\det(x)) = v(\mathrm{nr}_{M_n(D)/k}(x)).$$

*Proof.* Note that $v(\mathrm{nr}_{M_n(D)/k}(x)) = v(\det(\widehat{\mu}(x)))$, where $\widehat{\mu} : M_n(D) \hookrightarrow M_{mn}(W)$ is the embedding discussed earlier. Since both the reduced norm map and the Dieudonné determinant are multiplicative, and any invertible matrix is a product of elementary matrices, it suffices to prove the equality for elementary matrices. The claim is clear for row-addition and row-switching matrices. It remains to check the claim for row-multiplication matrices. Since both reduced norms and the determinant are multiplicative, it suffices to consider the diagonal matrix $d = \mathrm{diag}(y, 1, 1 \ldots, 1)$, for $y \in D$. Then $v_D(\det(d)) = v_D(y)$, and $v(\mathrm{nr}_{M_n(D)/k}(d)) = v(\mathrm{nr}_{D/k}(y))$. By Equation (2.1), the two valuations are equal. □

### 2.2.2. Orders in central simple algebras over number fields

Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$, and let $\Lambda$ be an $\mathcal{O}_K$-order in $A$. We denote by $K_\nu$ and $\mathcal{O}_\nu$ the completions of $K$, and respectively $\mathcal{O}_K$, at a place $\nu \in \mathrm{Pl}(K)$. Define $A_\nu := K_\nu \otimes_K A$ and $\Lambda_\nu := \mathcal{O}_\nu \otimes_R \Lambda$. If $\nu$ is an infinite place, we set $\mathcal{O}_\nu := K_\nu$ and $\Lambda_\nu := A_\nu$.

Consider the set of $\mathcal{O}_K$-orders in $A$ locally isomorphic to $\Lambda$, called the *genus* of $\Lambda$. By the Skolem-Noether theorem, this set consists of $\mathcal{O}_K$-orders $\Gamma \subset A$ such that $\Gamma_\nu = \xi_\nu \Lambda_\nu \xi_\nu^{-1}$ for some $\xi_\nu \in A_\nu^\times$ at all finite places $\nu$ of $K$. Local isomorphisms do not necessarily lift to global isomorphisms, and we wish to investigate the isomorphism classes in the genus of $\Lambda$. Again, by the Skolem-Noether theorem, $\Gamma$ and $\Lambda$ are isomorphic as $\mathcal{O}_K$-orders if they are conjugate, i.e. if $\Gamma = \xi \Lambda \xi^{-1}$ by some $\xi \in A^\times$.

Let $J_K$ be the ideles of $K$ and $J_A$ the ideles of $A$. We have reduced norm maps $\mathrm{nr}_{A/K} : A \to K$ and $\mathrm{nr}_{A_\nu/K_\nu} : A_\nu \to K_\nu$, which induce $\mathrm{nr} : J_A \to J_K$ where $\mathrm{nr}((a_\nu)_\nu) = (\mathrm{nr}_{A_\nu/K_\nu}(a_\nu))_\nu$. Denote the normalizer of $\Lambda_\nu$ by $\mathcal{N}(\Lambda_\nu) = \{\xi \in A_\nu^\times \mid \xi\Lambda_\nu\xi^{-1} = \Lambda_\nu\}$, and the restricted product $\prod'_\nu \mathcal{N}(\Lambda_\nu) := J_A \cap \prod_\nu \mathcal{N}(\Lambda_\nu)$.

The global isomorphism classes correspond to the double cosets $A^\times\backslash J_A/\prod'_\nu \mathcal{N}(\Lambda_\nu)$, whose cardinality we will denote by $G(\Lambda)$ and is known as the *type number* of $\Lambda$. Since $A$ is of degree $n \geq 3$ over a number field, it satisfies the Eichler condition. As a consequence of strong approximation, the reduced norm induces a bijection

$$\mathrm{nr} : A^\times\backslash J_A/\prod_\nu{}' \mathcal{N}(\Gamma_\nu) \to K^\times\backslash J_K/\,\mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Gamma_\nu)) = J_K/K^\times\,\mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Gamma_\nu)). \quad (2.3)$$

For more details, see Chapter 34 in [34] and Section 28.4 in [45]. Since the codomain has the structure of an abelian group, we can more easily perform the idelic calculations.

Section 2.3

## The building for $SL_r(k)$

Let $k$ be a non-archimedean local field with valuation ring $R$, maximal ideal $P$ and uniformizer $\pi$. Let $D$ be a central division algebra of degree $m$ over $k$, with unique maximal $R$-order $\Delta$, and prime element $\boldsymbol{\pi} \in \Delta$ such that $\boldsymbol{\pi}^m = \pi$. In this section, we construct the building for $SL_n(D)$ following Chapter 9 in [36].

Consider the free left $M_n(D)$-module $V := D^n$. A *lattice* in $V$ is a free (left) $\Delta$-submodule $L \subset V$ such that $D \otimes_\Delta L = V$. We call two lattices $L$ and $L'$ *homothetic* if there exists $a \in D^\times$ such that $L' = aL$. This is clearly an equivalence relation, and we will denote the homothety class of $L$ by $[L]$.

The building for $SL_n(D)$ can be equipped with the structure of a simplicial complex, defined as follows. Its vertices are the set of homothety classes $[L]$ of lattices. There is an edge between two vertices $x$ and $y$ if there exists a lattice $L$ in the class of $x$ and a lattice $L'$ in the class of $y$ such that

$$\boldsymbol{\pi} L \subsetneq L' \subsetneq L.$$

The $s$-simplices are given by *flags*

$$\boldsymbol{\pi} L_1 \subsetneq L_{s+1} \subsetneq \cdots \subsetneq L_2 \subsetneq L_1.$$

The maximal $(n-1)$-simplices are called *chambers*.

The group $GL_n(D)$ acts transitively on the set of $\Delta$-lattices in $D^n$, preserving homothety classes, and therefore acts on the building. Moreover, it also preserves adjacency relations, and left multiplication by $\xi \in GL_n(D)$ is therefore a simplicial map.

**Definition 2.3.1.** Let $g \in GL_n(D)$. We define the **type** of the matrix $g$ by

$$t(g) := v_D(\det(g)) \pmod{n},$$

where det is the Dieudonné determinant.

We extend types to vertices in the building. Start with some lattice $L_0$, and assign it type 0. We denote this by $t(L_0) \equiv 0 \pmod{n}$. Any other lattice in $D^n$ is of the form $\xi L_0$ for some $\xi \in GL_n(D)$, and we assign

$$t(\xi L_0) := t(\xi) \in \mathbb{Z}/n\mathbb{Z}.$$

This implies that for any $\zeta \in GL_n(D)$ and any lattice $L \subset D^n$,

$$t(\zeta L) \equiv t(\zeta) + t(L) \pmod{n}. \tag{2.4}$$

The type map is well-defined on homothety classes, since given any $a \in D^\times$, we have $t(aL) \equiv n \cdot v_D(a) + t(L) \equiv t(L) \pmod{n}$, and we will denote $t([L]) \equiv t(L)$. Therefore, we have a well-defined type function for any vertex in the building. In addition, the action of $SL_n(D)$ on the building is type-preserving.

Recall that $\boldsymbol{\pi}\Delta = \Delta\boldsymbol{\pi}$, so $\boldsymbol{\pi}^{m_1}\Delta f_1 \oplus \cdots \oplus \boldsymbol{\pi}^{m_n}\Delta f_n = \Delta\boldsymbol{\pi}^{m_1} f_1 \oplus \cdots \oplus \Delta\boldsymbol{\pi}^{m_n} f_n$ for any basis $\{f_1, \ldots, f_n\}$ of $V$. As described in [36, page 116], given a chamber represented by a flag $L_0 \supsetneq \cdots \supsetneq L_{n-1} \supsetneq \boldsymbol{\pi} L_0$, we can find a basis $e_1, \ldots, e_n$ for $D^n$ such that

$$L_0 = \Delta e_1 \oplus \Delta e_2 \oplus \cdots \oplus \Delta e_n$$
$$L_1 = \Delta\boldsymbol{\pi} e_1 \oplus \Delta e_2 \oplus \cdots \oplus \Delta e_n$$
$$\cdots$$
$$L_{n-1} = \Delta\boldsymbol{\pi} e_1 \oplus \Delta\boldsymbol{\pi} e_2 \oplus \cdots \oplus \Delta\boldsymbol{\pi} e_{n-1} \oplus \Delta e_n.$$

In particular note that the diagonal matrices $d_i = \mathrm{diag}(\underbrace{\boldsymbol{\pi}, \ldots, \boldsymbol{\pi}}_{i \text{ times}}, 1, \ldots, 1)$ give $L_i = d_i L_0$, so $t(L_i) = i$. Therefore, in any chamber, all vertices have distinct types.

Take a basis $\{e_1, \ldots, e_n\}$ of $D^n$. We define the **apartment** $\mathcal{A}$ to be the subcomplex of the building whose vertices correspond to homothety classes of lattices $\Delta\boldsymbol{\pi}^{m_1} e_1 \oplus \Delta\boldsymbol{\pi}^{m_2} e_2 \oplus \cdots \oplus \Delta\boldsymbol{\pi}^{m_n} e_n$, for $m_i \in \mathbb{Z}$. Such homothety classes are obtained by left multiplying the lattice $L_0 := \Delta e_1 \oplus \Delta e_2 \oplus \cdots \oplus \Delta e_n$ with the diagonal matrix $\mathrm{diag}(\boldsymbol{\pi}^{m_1}, \boldsymbol{\pi}^{m_2}, \ldots, \boldsymbol{\pi}^{m_n})$. Assigning $L_0$ type 0, we have

$$t(\Delta\boldsymbol{\pi}^{m_1}e_1 \oplus \Delta\boldsymbol{\pi}^{m_2}e_2 \oplus \cdots \oplus \Delta\boldsymbol{\pi}^{m_n}e_n) \equiv \sum_{i=1}^{n} m_i \pmod{n}.$$

*Remark* 2.3.2. To simplify our notation, we will denote the lattice

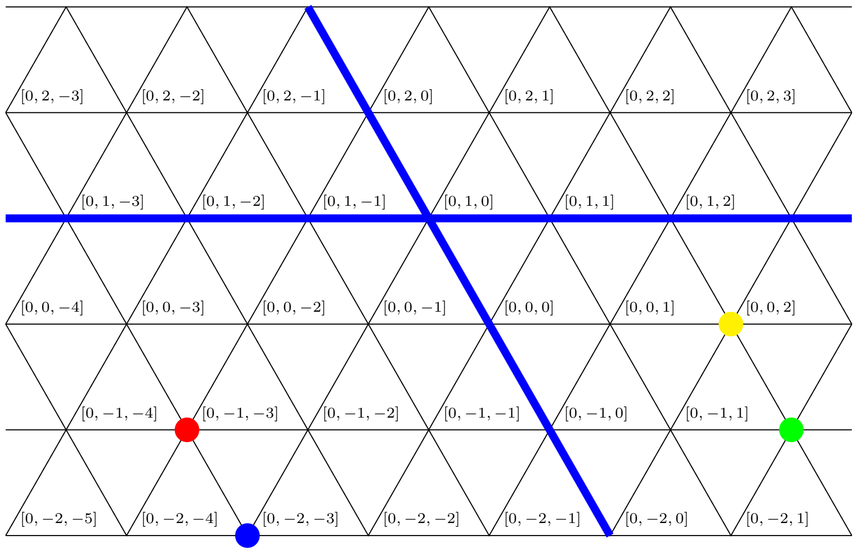
$$L = \Delta\boldsymbol{\pi}^{m_1}e_1 \oplus \Delta\boldsymbol{\pi}^{m_2}e_2 \oplus \cdots \oplus \Delta\boldsymbol{\pi}^{m_n}e_n$$

by the tuple $L = (m_1, m_2, \ldots, m_n)$, and the homothety class by $[L] = [m_1, \ldots, m_n]$. Two lattices given by the tuples $(m_1, \ldots, m_n)$ and $(\ell_1, \ldots, \ell_n)$ are homothetic if and only if there exists $r \in \mathbb{Z}$ such that $m_i - \ell_i = r$ for all $i \leq n$. In particular, $[m_1, \ldots, m_n] = [0, m_2 - m_1, \ldots, m_n - m_1]$, and we can identify each vertex in $\mathcal{A}$ by a unique $n$-tuple in $\mathbb{Z}^n$ whose first entry is 0.

Then $\mathcal{A}$ is a tessellation of $\mathbb{R}^{n-1}$ by chambers, where each vertex $[m_1, \ldots, m_n] = [0, m_2 - m_1, \ldots, m_n - m_1]$ lies on the hyperplanes $x_i - x_j = (m_i - m_1) - (m_j - m_1) = m_i - m_j$ where $i \neq j$. We will refer to the vertex $[0, 0, \ldots, 0]$ as the *origin* of the apartment $\mathcal{A}$, and the origin is at the intersection of all hyperplanes of the form $x_i - x_j = 0$ where $i \neq j$. In Figure 2.1, we see a piece of an apartment in the building for $SL_3(D)$, which is a tessellation of $\mathbb{R}^2$ by equilateral triangles. The emphasized lines correspond to the hyperplanes $x_1 - x_3 = 0$ and $x_1 - x_2 = -1$.

We have a few actions on the apartment $\mathcal{A}$, each of them given by left multiplication of the homothety class corresponding to a vertex by a monomial matrix. Monomial matrices, also known as generalized permutation matrices, are matrices with precisely one nonzero entry in each row and column. There are two kinds of

Figure 2.1: Actions on an apartment.




actions on the apartment that we will often use. The first kind of action is induced by left-multiplication by diagonal matrices, which act as a translation on the apartment.

**Lemma 2.3.3.** *The diagonal matrix diag*$(\boldsymbol{\pi}^{a_1}, \boldsymbol{\pi}^{a_2}, \ldots, \boldsymbol{\pi}^{a_n})$ *induces an action on the apartment by translating each vertex* $[L] = [m_1, \ldots, m_n] \mapsto [m_1 + a_1, \ldots, m_n + a_n]$.

In Figure 2.1, multiplication by $\text{diag}(1, \boldsymbol{\pi}, \boldsymbol{\pi}^5)$ induces a translation of the apartment, where the red vertex is translated to the yellow vertex, and the blue vertex is translated to the green vertex.

The second kind of action is given by reflections with respect to hyperplanes in the apartment, which as we will see are type-preserving:

**Lemma 2.3.4.** *Fix an apartment in the building for* $SL_n(D)$*, and let* $H$ *be the hyperplane in the apartment given by the equation* $x_r - x_s = a$*. Then the matrix* $\xi_{r,s,a}$ *defined by*

$$(\xi_{r,s,a})_{ij} = (\boldsymbol{\pi}^{\beta_i} \delta_{\sigma(i)j})$$

*where $\delta_{ij}$ is the Kronecker delta, $\sigma = (rs)$ and*

$$\beta_i = \begin{cases} 0 & \textit{if both } i \neq r \textit{ and } i \neq s \\ a & \textit{if } i = r \\ -a & \textit{if } i = s \end{cases}$$

*corresponds to the reflection with respect to $H$. In addition, the action on the apartment is type-preserving.*

*Proof.* A reflection with respect to any hyperplane is a composition of two transformations: a reflection with respect to the hyperplane going through the origin $x_r - x_s = 0$, and a translation. The reflection with respect to $x_r - x_s = 0$ is given by the elementary matrix $e_{r,s}$, obtained by interchanging the $r$th and $s$th rows in the identity matrix. The matrix for the translation is a diagonal matrix, whose entries are determined by the image of the origin $[0, 0, \ldots, 0]$ under the reflection with respect to $H$. Since the origin gets mapped to $[0, \ldots, a, \ldots, -a, \ldots, 0]$, where $a$ is in the $r$ position, and $-a$ in the $s$ position, this translation corresponds to the diagonal matrix $d_{r,s,a} = (\boldsymbol{\pi}^{\beta_i}\delta_{ij})$, where

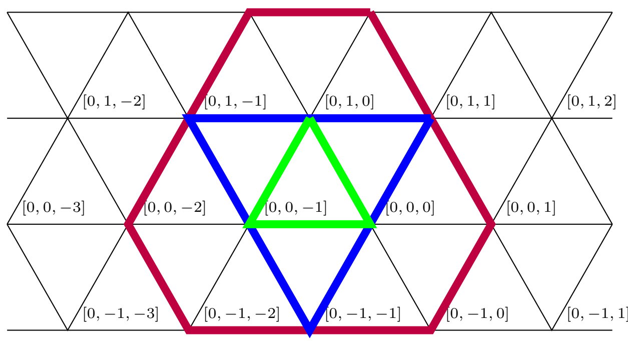
$$\beta_i = \begin{cases} 0 & i \neq r \text{ and } i \neq s \\ a & i = r \\ -a & i = s. \end{cases}$$

Then $\xi_{r,s,\mu} = d_{r,s,\mu} \cdot e_{r,s} = (\boldsymbol{\pi}^{\beta_i}\delta_{\sigma(i)j})$, and indeed the action is type-preserving since $t(\eta_{r,s,\mu}) = a - a = 0$. □

In Figure 2.1, the reflection with respect to the hyperplane $x_1 - x_3 = 0$ is given by the permutation matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, and the reflection with respect to the hyperplane $x_1 - x_2 = -1$ by the monomial matrix $\begin{pmatrix} 0 & \boldsymbol{\pi}^{-1} & 0 \\ \boldsymbol{\pi} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Products of these actions create more intricate actions, such as rotations in the apartment. In Figure 2.2, the monomial matrix $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \boldsymbol{\pi}^{-1} & 0 \end{pmatrix}$ induces a clockwise rotation of the outlined polytopes.

Figure 2.2:




We make the following observation about the action of monomial matrices with type 0:

**Lemma 2.3.5.** *Any monomial matrix of type* 0 *acts as a product of reflections on the apartment* $\mathcal{A}$.

*Proof.* Any monomial matrix decomposes as a product of a diagonal and a permutation matrix. Permutation matrices correspond to products of reflections with respect to hyperplanes going through the origin, so it suffices to prove the claim for diagonal matrices. In addition, $GL_n(D)$ acts simplicially on the building, so it suffices to show any diagonal matrix of type 0 will act as a product of reflection on some chamber.

By Lemma 2.3.3, any diagonal matrix induces a translation on the apartment. Fix a chamber $C$ in $\mathcal{A}$, and denote the vertices of $C$ by $[L_i]$. Let $C'$ be the translation of $C$ under $d$, with vertices $[L_i'] = d[L_i]$. Since we assume that $t(d) = 0$, it follows that $t(L_i) = t(L_i')$ for all $i$.

We have an additional action on the apartment sending $C$ to $C'$. In particular, in any building, one can connect any two chambers by a product of reflections (see Example 4.3.4). Let $\xi := \xi_{r_1,s_1,a_1}\xi_{r_2,s_2,a_2}\dots\xi_{r_\ell,s_\ell,a_\ell}$ correspond to such a product of reflections sending $C$ to $C'$, and let $[L_i''] = \xi[L_i]$. Then $t(\xi) = 0$, and $t(L_i) = t(L_i'')$.

Since all the vertices in a chamber have distinct types, so it must be true that $[L_i'] = [L_i'']$ for all $i \leq n$, so $\xi$ and $d$ act the same on the chamber $C$ in $\mathcal{A}$. Therefore, $\xi$ and $d$ must give the same action on $\mathcal{A}$, and we have proven our claim. $\square$

We have a connection between orders in the algebra $M_n(D)$ and vertices in the building. Recall the lattice $L_0 = \Delta e_1 \oplus \cdots \oplus \Delta e_n$. The ring of linear transformations $\mathrm{End}_\Delta(L_0)$ is a maximal order, which we can identify with $M_n(\Delta)$ (for more details, see [34, page 170]). By [34, (17.4)], any other maximal order $\Lambda$ is of the form $\xi M_n(\Delta)\xi^{-1} = \mathrm{End}_\Delta(\xi L_0)$ for some $\xi \in GL_n(D)$. Moreover, by embedding $D^\times \hookrightarrow GL_n(D)$ diagonally, it follows that for any $a \in D^\times$, we have $\mathrm{End}_\Delta(aL_0) = aM_n(\Delta)a^{-1} = M_n(a\Delta a^{-1}) = M_n(\Delta)$, since $\Delta$ is the unique maximal $R$-order in $D$, so we have a correspondence

$$\text{vertices in the building for } SL_n(D)$$
$$\updownarrow$$
$$\text{homothety classes of lattices } [L] \text{ in } D^n$$
$$\updownarrow$$
$$\text{maximal orders in } M_n(D).$$

This correspondence will allow us to investigate some special classes of orders in $M_n(D)$, as shown in the following example.

**Example 2.3.6.** We construct the Bruhat-Tits tree for $SL_2(\mathbb{Q}_p)$, which is the building in th case $n = 2$. Fix a basis $\{e_1, e_2\}$ and identify $M_2(\mathbb{Z}_p)$ with the homothety class of the lattice $L_0 = \mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2$. The building for $SL_2(\mathbb{Q}_p)$ is a $(p+1)$-regular tree, where the neighbors of a vertex $[L]$ are given by the homothety classes $[\xi L]$, for $\xi \in \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}, a \in \mathbb{Z}/p\mathbb{Z} \right\}$.

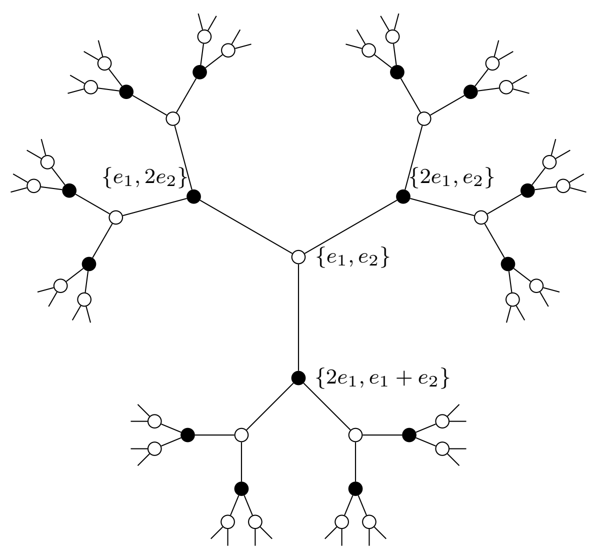
For example, for $p = 2$, the three neighbors of $L_0 = \mathbb{Z}_2 e_1 \oplus \mathbb{Z}_2 e_2$ are

$$\mathbb{Z}_2(2e_1) \oplus \mathbb{Z}_2 e_2$$

$$\mathbb{Z}_2 e_1 \oplus \mathbb{Z}_2(2e_1 + 2e_2) = \mathbb{Z}_2 e_1 \oplus \mathbb{Z}_2(2e_2)$$

$$\mathbb{Z}_2(2e_1) \oplus \mathbb{Z}_2(e_1 + e_2).$$

Using rather ad-hoc notation (which we won't use in the rest of the thesis), we denote them by $\{2e_1, e_2\}$, $\{e_1, 2e_2\}$ and $\{2e_1, e_1+e_2\}$. We can get subsequent neighbors inductively.



We say an order $\Gamma \subset M_2(\mathbb{Q}_p)$ is an *Eichler* order if it can be written as the intersection of two maximal orders. Consider the Eichler order

$$\Gamma = \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \cap \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ 2^{-1}\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}.$$

Since the maximal order $\begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ 2^{-1}\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \cong \mathrm{End}_{\mathbb{Z}_2}(\mathbb{Z}_2 2e_1 \oplus \mathbb{Z}_2 e_2)$, we can associate to $\Gamma$ the path in the tree from $\{e_1, e_2\}$ to $\{2e_1, e_2\}$. Let $\omega = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, then

$$\omega\Gamma\omega^{-1} = \omega \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \omega^{-1} \cap \omega \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ 2^{-1}\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \omega^{-1} = \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ 2^{-1}\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \cap \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} = \Gamma.$$

More generally, any element of the normalizer $\mathcal{N}(\Gamma) = \{\xi \in GL_2(\mathbb{Q}) : \xi\Gamma\xi^{-1} = \Gamma\}$ will act on the path by either fixing it, or swapping the two extremal vertices.

# Chapter 3

# Tiled orders and their convex polytopes

The class of orders which we investigate in this work is motivated by previous calculations of type numbers of Eichler orders in quaternion algebras, such as the works of Eichler [8], Peters [30], Pizer [31], and Vignéras [44]. In particular, for a (global) Eichler order $\Gamma$, one important ingredient for computing type numbers are the normalizers of local orders $\Gamma_\nu$. As seen in Example (2.3.6), one can associate to each completion $\Gamma_\nu$, a path in the tree for $SL_2(D_\nu)$, on which the normalizer of $\Gamma_\nu$ will act by interchanging the extremal vertices.

Based on the combinatorial description of (local) Eichler orders from Example (2.3.6), we introduce a suitable generalization for orders in central simple algebras of degree $n \geq 3$ over non-archimedean local fields and study some of their arithmetic and algebraic properties. The generalization we introduce yields what we will call *tiled* orders, which are known under various names, and have been studied in many contexts.

Let us establish some notation. We consider the central simple $k$-algebra $B := M_n(D)$, where $D$ is a central division algebra of degree $m$ over a non-archimedean local field $k$. We denote the valuation ring of $k$ by $R$, which has a unique maximal ideal $P$ and uniformizer $\pi$, and we denote by $\Delta$ the unique maximal $R$-order in $D$, with a prime element $\boldsymbol{\pi} \in \Delta$ such that $\boldsymbol{\pi}^m = \pi$, and by $\mathfrak{p} := \boldsymbol{\pi}\Delta = \Delta\boldsymbol{\pi}$ the unique two-sided ideal of $\Delta$ (see Section 14 in [34]). We pick a basis $\{e_1, \ldots, e_n\}$ of $D^n$ as a free left $D$-module, and fix the apartment $\mathcal{A}$ in the building for $SL_n(D)$ determined by this basis.

We will consider orders of the form $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ in $M_n(D)$, where $\mu_{ij} \in \mathbb{Z}$ and $\mu_{ii} = 0$ for all $i \leq n$. In [39], Shemanske associates to each such tiled order a convex polytope $C_\Gamma$ in the apartment $\mathcal{A}$. It turns out, we can study the order $\Gamma$, which is an algebraic object, by studying $C_\Gamma$, which is mainly a combinatorial construction.

To study properties of the polytope $C_\Gamma$, we embark on investigating structural invariants $m_{ij\ell} := \mu_{ij} + \mu_{j\ell} - \mu_{i\ell} \geq 0$. Introduced by Zassenhaus in [47], they determine the isomorphism class of $\Gamma$. In fact, they also determine geometric data for the associated polytope $C_\Gamma$, and in particular the congruence class of $C_\Gamma$. This allows us to determine algebraic properties for a given isomorphism class of tiled orders by examining geometric and combinatorial properties of the corresponding congruence class of polytopes.

In Sections 3.4-3.6, we study how certain algebraic properties of tiled orders manifest themselves in the associated polytopes. In particular, by Proposition 3.4.3, an order $\Gamma \subset M_n(D)$ is hereditary if and only if it is tiled and its polytope is a simplex in the building for $SL_n(D)$. In addition, every order in $M_n(D)$ has a canonical chain of overorders, called the radical idealizer chain terminating in a hereditary order.

Proposition 3.5.6 gives a geometric interpretation for each of the overorders in the chain.

After this brief detour, we return to investigate isomorphisms of tiled orders in terms of actions on the associated polytopes. Let $\mathcal{N}(\Gamma) = \{\xi \in GL_n(D) : \xi\Gamma\xi^{-1} = \Gamma\}$ be the normalizer of $\Gamma$ in $GL_n(D)$. In [40], Shemanske shows that one may study the normalizer $\mathcal{N}(\Gamma) = \{\xi \in GL_n(D) : \xi\Gamma\xi^{-1} = \Gamma\}$ by studying the symmetries of $C_\Gamma$. In Section 2.8, we consider the case $\Gamma \subseteq M_n(k)$; Haefner and Pappacena prove in [15] that $R$-automorphisms of $\Gamma$ give automorphisms of an associated multigraph $Q(\Gamma)$, which we will call quivers. In Theorem 3.9.3, we associate to each $\Gamma$ a tiled order $\Gamma_c$, obtained by scaling the polytope $C_\Gamma$ by $n$ and then translating it in a convenient manner. Its polytope has the same symmetries as $C_\Gamma$, which also induce automorphisms of an associated valued quiver $Q_v(\Gamma_c)$. In addition, the exponent matrix of $\Gamma_c$ makes the symmetries more transparent. The main results in the chapter can be summarized in the following theorem, which is a combination of Proposition 3.7.8 and Theorem 3.9.3.

**Theorem.** *Consider a tiled order* $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ *in* $M_n(D)$ *with structural invariants* $m_{ij\ell}$.

*(a) Let* $H := \{\sigma \in S_n : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ *for all* $i, j, \ell \leq n\}$. *Then* $\mathcal{N}(\Gamma) = \bigcup_{\sigma \in H} \xi_\sigma D^\times \Gamma^\times$, *where* $\xi_\sigma := (\boldsymbol{\pi}^{\mu_{i1} - \mu_{\sigma(i)\sigma(1)}})$.

*(b) Let* $\Gamma_c := (\mathfrak{p}^{\nu_{ij}})$ *be the centered tiled order associated to* $\Gamma$, *where* $\nu_{ij} := \sum_{\ell=1}^{n} m_{ij\ell}$. *Then* $H = \{\sigma \in S_n : \nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ *for all* $i, j \leq n\}$.

*(c) When* $\Gamma \subseteq M_n(k)$, *there is a group isomorphism between* $H$ *and the automorphisms of the valued quiver* $Q_v(\Gamma_c)$.

Since each $\sigma \in H$ gives a monomial matrix $\xi_\sigma$ which induces an automorphism of $\Gamma$ and acts by rigid motions on $C_\Gamma$, we refer to them as the "symmetries of $C_\Gamma$". Theorem 3 has several corollaries describing normalizers of tiled orders. In particular, by Corollary 3.9.7, if $\Gamma$ is hereditary and its polytope is a chamber, then $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong \mathbb{Z}/n\mathbb{Z}$. Finally, Corollary 3.10.1 states that 2-transitive proper subgroups of $S_n$ are not eligible as symmetry groups of convex polytopes.

Section 3.1

## Definitions

We first give a geometric definition of a tiled order. Consider the free left $D$-module $D^n$. Construct the building for $SL_n(D)$ as in Chapter 2, where the vertices of the building are in correspondence to the set of homothety classes $[L]$ of free left $\Delta$-lattices in $D^n$, and therefore with the set of maximal orders $\mathrm{End}_\Delta(L)$. Take a basis $\{e_1, \ldots, e_n\}$ of $D^n$ and let $\mathcal{A}$ be the apartment in the building for $SL_n(D)$ whose vertices are homothety classes $[L]$, where $L = \boldsymbol{\pi}^{m_1}\Delta e_1 \oplus \boldsymbol{\pi}^{m_2}\Delta e_2 \oplus \cdots \oplus \boldsymbol{\pi}^{m_n}\Delta e_n = \Delta\boldsymbol{\pi}^{m_1} e_1 \oplus \Delta\boldsymbol{\pi}^{m_2} e_2 \oplus \cdots \oplus \Delta\boldsymbol{\pi}^{m_n} e_n$ is a free left $\Delta$-lattice. We denote the lattice $L$ by $L = (m_1, m_2, \ldots, m_n)$, and the homothety class by $[L] = [m_1, m_2, \ldots, m_n] = [0, m_2 - m_1, \ldots, m_n - m_1]$.

**Definition 3.1.1.** We say an order $\Gamma \subseteq M_n(D)$ is *tiled* if $\Gamma = \bigcap_{i=1}^r \mathrm{End}_\Delta(L_i)$ is the intersection of finitely many maximal orders $\Lambda_i := \mathrm{End}_\Delta(L_i)$, whose corresponding vertices $[L_i]$ are in a fixed apartment $\mathcal{A}$.

Let $L_0 = \Delta e_1 \oplus \cdots \oplus \Delta e_n$ be denoted by $(0, 0, \ldots, 0)$, and assign it type $t(L_0) = 0$. Identify the vertex $[L_0] = [0, 0, \ldots, 0]$ with $M_n(\Delta) = \mathrm{End}_\Delta(L_0)$. Then any other vertex in the apartment $\mathcal{A}$ corresponds to a homothety class $[\xi L_0] = [m_1, m_2, \ldots, m_n]$

where $\xi = (\boldsymbol{\pi}^{m_i}\delta_{ij}) \in GL_n(D)$ is a diagonal matrix. The maximal order corresponding to $[\xi L_0] = [m_1, \ldots, m_n]$ is given by

$$\xi M_n(\Delta)\xi^{-1} = \begin{pmatrix} \Delta & \mathfrak{p}^{m_1-m_2} & \cdots & \mathfrak{p}^{m_1-m_n} \\ \mathfrak{p}^{m_2-m_1} & \Delta & \cdots & \mathfrak{p}^{m_2-m_n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{p}^{m_n-m_1} & \mathfrak{p}^{m_n-m_2} & \cdots & \Delta \end{pmatrix}.$$

More generally, if $\Gamma = \cap_r \Lambda_r$, $r \in \mathbb{Z}_{>0}$, is the intersection of finitely many maximal orders where $\Lambda_r = (\mathfrak{p}^{m_i^{(r)} - m_j^{(r)}})$, we get $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ for $\mu_{ij} = \max_r(m_i^{(r)} - m_j^{(r)})$. Note that

$$\mu_{ii} = 0, \qquad \text{for all} \quad i \leq n.$$

Since $\Gamma$ is an intersection of finitely many orders, it is an order, and therefore closed under multiplication. This forces

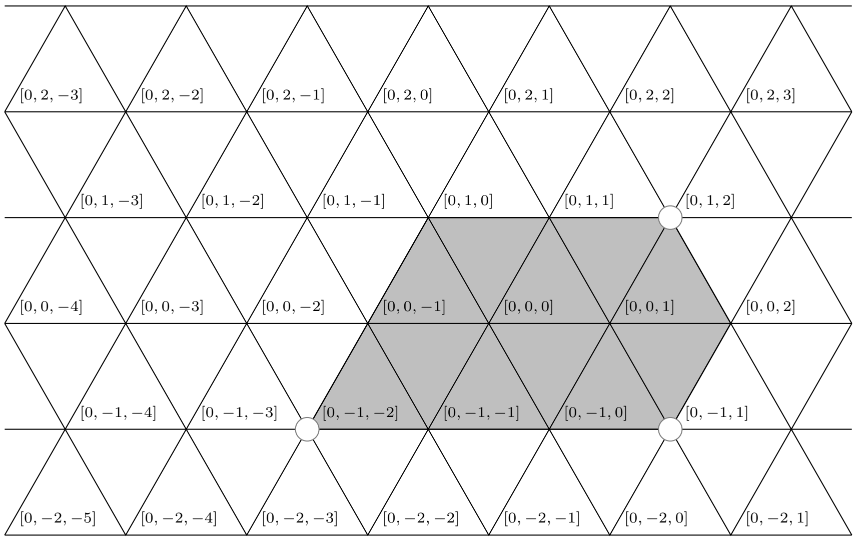
$$\mu_{ij} + \mu_{j\ell} \geq \mu_{i\ell}, \qquad \text{for all} \quad i, j, \ell \leq n. \tag{3.1}$$

**Definition 3.1.2.** Let $S = (\mathfrak{p}^{m_{ij}}) \subseteq M_n(D)$ be the subset of matrices where the $(i, j)$th entry belongs to $\mathfrak{p}^{\mu_{ij}}$ for any $\mu_{ij} \in \mathbb{Z}$. We denote by $M_S = (m_{ij})$ the *exponent matrix* of $S$.

In particular, we will often use the exponent matrix $M_\Gamma$ of a tiled order $\Gamma$, but the definition above applies to subsets that are not necessarily orders in $M_n(D)$.

**Example 3.1.3.** Recall the lattice $L_0 = (0, 0, \ldots, 0)$ corresponding to the maximal order $M_n(\Delta)$. Take $n = 3$, and let $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ be the maximal orders $\Lambda_1 = \xi_1 M_3(\Delta)\xi_1^{-1} = \mathrm{End}_\Delta(\xi_1 L_0)$ for $\xi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{\pi} & 0 \\ 0 & 0 & \boldsymbol{\pi}^2 \end{pmatrix}$, $\Lambda_2 = \xi_2 M_3(\Delta)\xi_2^{-1} =$

Figure 3.1:




$\mathrm{End}_\Delta(\xi_2 L_0)$ for $\xi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{\pi}^{-1} & 0 \\ 0 & 0 & \boldsymbol{\pi} \end{pmatrix}$, and $\Lambda_3 = \xi_3 M_3(\Delta)\xi_3^{-1} = \mathrm{End}_\Delta(\xi_3 L_0)$ for $\xi_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{\pi}^{-1} & 0 \\ 0 & 0 & \boldsymbol{\pi}^{-2} \end{pmatrix}$. The corresponding exponent matrices are given by

$$M_{\Lambda_1} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \qquad M_{\Lambda_2} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} \qquad M_{\Lambda_3} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{pmatrix},$$

and the vertices emphasized in Figure 3.1 then correspond to the homothety classes $[L_1], [L_2]$ and $[L_3]$ and therefore to the maximal orders $\Lambda_1, \Lambda_2$ and $\Lambda_3$.

Then $\Gamma = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3$ has exponent matrix

$$M_\Gamma = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

There are a few equivalent definitions of tiled orders, which we will use in proofs depending on our purposes.

**Proposition 3.1.4** (Plesken [33], Shemanske [39], Tarsy [41])**.** *Let $\Gamma \subseteq M_n(D)$ be an order. The following statements are equivalent:*

*(a)* $\Gamma$ *is tiled.*

*(b)* $\Gamma$ *is an intersection of finitely many maximal orders, whose corresponding vertices lie in a fixed apartment.*

*(c)* $\Gamma$ *contains a conjugate of the diagonal ring* $\begin{pmatrix} \Delta & & & \\ & \Delta & & \\ & & \ddots & \\ & & & \Delta \end{pmatrix}$.

*(d)* $\Gamma$ *contains a set of primitive orthogonal idempotents* $\epsilon_1, \dots, \epsilon_n$ *with* $\epsilon_1 + \cdots + \epsilon_n = 1 \in M_n(D)$*, such that* $\epsilon_i \Gamma \epsilon_i$ *is a maximal order in* $\epsilon_i M_n(D) \epsilon_i$ *for* $i = 1, \dots, n$.

*(e)* $\Gamma$ *is conjugate to an order of the form* $(\mathfrak{p}^{\mu_{ij}})$ *with*

$$\mu_{ii} = 0 \qquad \textit{for all} \qquad i \leq n$$

$$\mu_{ij} + \mu_{j\ell} \geq \mu_{i\ell} \qquad \textit{for all} \quad i, j, \ell \leq n.$$

Most of the results in this thesis consider tiled orders of the form $(\mathfrak{p}^{\mu_{ij}})$ with $\mu_{ij}$ as described above, and therefore implicitly assume the tiled order is an intersection of maximal orders $\bigcap_{i=1}^r \mathrm{End}_\Delta(L_i)$, where the vertices/homothety classes $[L_i]$ lie in the fixed apartment $\mathcal{A}$. In any case, by Proposition 3.1.4, any tiled order $\Gamma$ is conjugate to an order of this form, so up to a change of apartment, we may always assume $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ as in the proposition above.

Now that we have an algebraic characterization of tiled orders, let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a tiled order. The hyperplanes $H_{ij} := x_i - x_j = \mu_{ij}$ bound a convex region in $\mathbb{R}^{n-1}$. In [39], Shemanske defines $C_\Gamma$ to be the convex polytope determined by the inequalities $-\mu_{ji} \le x_i - x_j \le \mu_{ij}$. Let $\overline{C_\Gamma}$ be the closure in $\mathbb{R}^{n-1}$ of $C_\Gamma$. In [39], it is shown that $\Gamma$ is the intersection of all the maximal orders whose vertices lie on $\overline{C_\Gamma}$, and that $\overline{C_\Gamma}$ is the convex hull of any set of vertices $[L_i]$ in the apartment such that $\Gamma = \cap_i \text{End}_\Delta(L_i)$.

In Example 3.1.3, $C_\Gamma$ is determined by the inequalities

$$-1 \le x_1 - x_2 \le 1$$
$$-2 \le x_1 - x_3 \le 2$$
$$-2 \le x_2 - x_3 \le 1.$$

Then $\overline{C_\Gamma}$ given by the shaded region in Figure 3.1, and corresponds to the convex hull of the lattices given by the columns of $M_\Gamma$. We can check that any maximal order whose corresponding vertex is on $\overline{C_\Gamma}$ contains $\Gamma$; for example $M_3(\Delta)$ is such a maximal order, corresponding to the vertex $[0, 0, \ldots, 0]$.

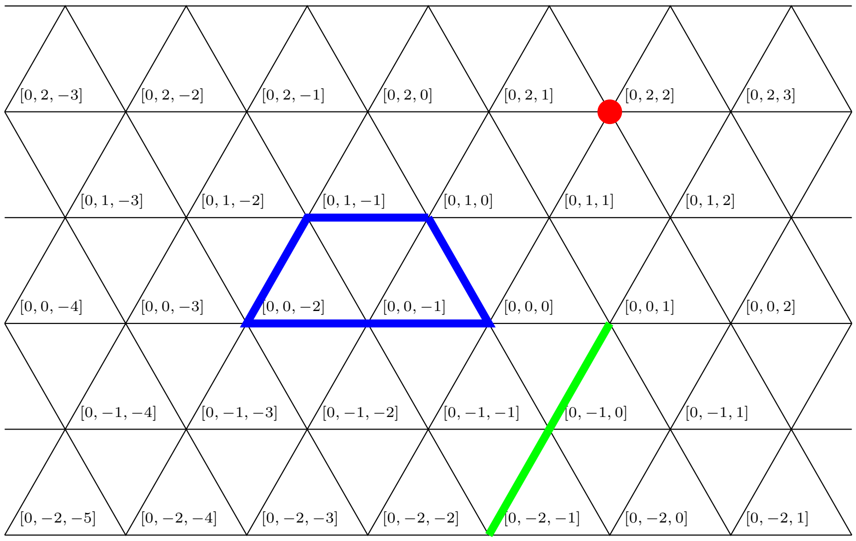
**Definition 3.1.5.** Let $\Gamma$ be a tiled order. We say $\Gamma$ has **full geometric rank** if $\overline{C_\Gamma}$ is $(n-1)$-dimensional in $\mathbb{R}^{n-1}$.

**Example 3.1.6.** Consider the tiled orders $\Gamma_1, \Gamma_2$ and $\Gamma_3$ with exponent matrices

$$M_{\Gamma_1} = \begin{pmatrix} 0 & -2 & -2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \qquad M_{\Gamma_2} = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix} \qquad M_{\Gamma_2} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The polytopes are shown in Figure 3.2, where $C_{\Gamma_1}$ is in red, $C_{\Gamma_2}$ in green, and $C_{\Gamma_3}$ in blue. Of the three, only $\Gamma_3$ has full geometric rank, while $\Gamma_1$ and $\Gamma_2$ do not.

Figure 3.2:




**Lemma 3.1.7.** $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ *is has full geometric rank if and only if* $\mu_{ij} + \mu_{ji} > 0$ *for all* $i \neq j$ *and* $i, j \leq n$.

*Proof.* $\overline{C_\Gamma}$ is $(n-1)$-dimensional if and only if all the pairs of bounding hyperplanes have distinct opposite hyperplanes $H_{ij} \neq H_{ji}$, if and only if $\mu_{ij} \neq -\mu_{ji}$ for all $i \neq j$. □

Section 3.2

## Distinguished vertices

We have seen in the previous section that $\overline{C_\Gamma}$ may contain many vertices, but we do not need to know all of them to reconstruct the tiled order or its polytope. There is a set of vertices that will be of importance to us, described in Remark II.4 in [33].

**Proposition 3.2.1** (Plesken, [33])**.** *Let* $\Gamma = (\mathfrak{p}^{\mu_{ij}}) \subseteq M_n(D)$ *be a tiled order. Recall the notation from Remark 2.3.2.*

(a) *The set of all nonzero* $\Gamma$*-lattices in* $D^n$ *is given by all*

$$L = (m_1, m_2, \ldots, m_n) \quad \textit{with} \quad m_i - m_j \leq \mu_{ij} \quad \textit{for all} \quad 1 \leq i, j \leq n.$$

(b) *Any two* $\Gamma$*-lattices* $L_1 = (m_1, \ldots, m_n)$ *and* $L_2 = (l_1, \ldots, l_n)$ *are isomorphic if and only if there exists an integer* $t \in \mathbb{Z}$ *such that* $m_i - l_i = t$ *for all* $i \leq n$*, that is, when* $L_1$ *and* $L_2$ *are in the same homothety class.*

(c) *Each projective indecomposable* $\Gamma$*-lattice is irreducible and isomorphic to* $P_\ell = (\mu_{1\ell}, \mu_{2\ell}, \ldots, \mu_{n\ell})$*, where* $P_\ell$ *is given by the* $\ell$*-th column of* $M_\Gamma$*.*

(d) *Similarly, each injective indecomposable* $\Gamma$*-lattice is irreducible and isomorphic to* $R_\ell = (-\mu_{\ell 1}, -\mu_{\ell 2}, \ldots, -\mu_{\ell n})$ *given by the* $\ell$*-th row of* $-M_\Gamma$*.*

In fact, each $\Gamma$-lattice in $D^n$ described above is indecomposable. Suppose we have $L = L_1 \oplus L_2$ for some $\Gamma$-sublattices $L_1, L_2 \subseteq L$. Then $D \otimes_\Delta L = (D \otimes_\Delta L_1) \oplus (D \otimes_\Delta L_2)$ is an $M_n(D) = (D \otimes_\Delta \Gamma)$-module in $D^n$. But $D \otimes_\Delta L = D^n$ is an irreducible $M_n(D)$-module, so either $D \otimes_\Delta L_1 = \{0\}$ or $D \otimes_\Delta L_2 = \{0\}$. Since $\Delta$ and $D$ are indecomposable and torsion-free, this means either $L_1 = \{0\}$ or $L_2 = \{0\}$

Since $C_\Gamma$ is bounded by the hyperplanes $x_i - x_j = \mu_{ij}$, statement (a) from Proposition 3.2.1 implies that each $\Gamma$-lattice in $D^n$ corresponds to a vertex on $\overline{C_\Gamma}$. Can we say more about the vertices corresponding to classes of projective and injective indecomposable $\Gamma$-lattices?

Consider the homothety classes and their corresponding vertices

$$[P_\ell] = [\mu_{1\ell}, \mu_{2\ell}, \ldots, \mu_{n\ell}] = [0, \mu_{2\ell} - \mu_{1\ell}, \ldots, \mu_{n\ell} - \mu_{1\ell}], \quad \text{for} \quad 1 \leq \ell \leq n.$$

defined in Proposition 3.2.1.

**Proposition 3.2.2** (Shemanske, [40])**.** *The vertices $[P_1], [P_2], \ldots, [P_n]$ defined above are extremal points on $C_\Gamma$, and they uniquely determine $C_\Gamma$ (and therefore $\Gamma$).*

*Proof.* We follow the proof from Proposition 2.2. in [40]. First, we need each $[P_\ell]$ to lie on the polytope. This amounts to checking that its $i^{\text{th}}$ and $j^{\text{th}}$ entries satisfy $-\mu_{ij} \le x_i - x_j \le \mu_{ij}$. Since

$$\mu_{ji} + \mu_{i\ell} \ge \mu_{j\ell}$$

$$\mu_{ij} + \mu_{j\ell} \ge \mu_{i\ell}$$

it follows that indeed

$$-\mu_{ji} \le \mu_{i\ell} - \mu_{j\ell} \le \mu_{ij}.$$

Next, we need each $[P_\ell]$ to be at the intersection of bounding hyperplanes for $C_\Gamma$. In fact, we claim that $[P_\ell]$ is at the intersection of the (affine) hyperplanes $\bigcap_{i\neq\ell} H_{i\ell}$, where $H_{i\ell}$ is a bounding hyperplane defined by $x_i - x_\ell = \mu_{i\ell}$ when $i \neq \ell$. Recall that $[P_\ell] = [\mu_{1\ell}, \mu_{2\ell}, \ldots, \mu_{n\ell}]$, so $[P_\ell]$ lies on each of the hyperplanes $x_i - x_\ell = \mu_{i\ell} - \mu_{\ell\ell} = \mu_{i\ell}$, and each $[P_\ell]$ is therefore an extremal vertex of $C_\Gamma$.

Arranging $P_1, \ldots, P_n$ as the columns of $M_\Gamma$, we see that these vertices give the bounds which determine $C_\Gamma$. Since each convex polytope uniquely determines a tiled order $\Gamma$, the $[P_\ell]$'s determine $\Gamma$ as well. $\square$

As we have seen, the set of vertices $[P_\ell]$ is quite special. In addition, the corresponding maximal orders allow us to reconstruct $\Gamma$.

**Lemma 3.2.3** (Shemanske, [40]). *Let* $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ *be a tiled order with convex polytope* $C_\Gamma$*. Then* $\Gamma = \cap_{\ell=1}^{n} \Lambda_\ell$*, where the* $\Lambda_\ell$ *are the maximal orders corresponding to the vertices* $[P_\ell]$ *of* $C_\Gamma$*.*

*Proof.* The proof follows from that of Proposition 2.2. in [40]. Each of the distinguished vertices has corresponding maximal order $\Lambda_\ell = (\mathfrak{p}^{\mu_{i\ell}-\mu_{j\ell}})$. Since $\mu_{i\ell}-\mu_{j\ell} \leq \mu_{ij}$ by Equation (3.1), we see that $\max_\ell(\mu_{i\ell} - \mu_{j\ell}) = \mu_{ij}$ by setting $\ell = j$. Therefore, $\cap_{\ell=1}^{n} \Lambda_\ell = (\mathfrak{p}^{\mu_{ij}}) = \Gamma$. □

From now on, we will refer to the vertices $[P_\ell]$ as the *distinguished vertices* of $C_\Gamma$.

*Remark* 3.2.4. We get similar results if instead we consider the homothety classes and the vertices given by

$$[R_\ell] = [-\mu_{\ell 1}, -\mu_{\ell 2}, \ldots, -\mu_{\ell n}] = [0, \mu_{\ell 1} - \mu_{\ell 2}, \ldots, \mu_{\ell 1} - \mu_{\ell n}] \quad \text{for} \quad 1 \leq \ell \leq n,$$

defined in Proposition 3.2.1. These vertices are also extremal and each $[R_\ell]$ is at the intersection of the hyperplanes $\cap_{j\neq\ell} H_{\ell j}$ for $j \neq \ell$ (therefore, in a way, opposite to the vertices $[P_\ell]$), uniquely determine $C_\Gamma$ (and $\Gamma$), and the intersection of their corresponding maximal orders is equal to $\Gamma$. We will touch more upon this duality between the two sets of vertices later in the chapter.

Shemanske showed in [39] that $\Gamma$ is the intersection of all the maximal orders whose vertices lie on and in $C_\Gamma$, and that there is a one-to-one correspondence between convex polytopes in the apartment and tiled orders associated to the apartment. It makes sense to investigate which algebraic properties of the order can be interpreted in terms of properties of its corresponding polytope, especially since as we shall see next, isomorphisms of tiled orders can be interpreted as actions on the polytope by rigid motions.

Let $\{e_{ii}\}_{i=1}^{n}$ be the set of orthogonal primitive idempotents given by the matrices with a one in the $(i,i)$ position, and zeros everywhere else. Since any tiled order $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ contains the above complete set of orthogonal idempotents, and $e_{ii}\Gamma e_{ii} \cong \Delta$ is a local ring, tiled orders are *semiperfect rings* (see [26]). Proposition 3 on p.77 in [23] states the following result.

**Lemma 3.2.5** (Lambek, [23]). *Let $S$ be a semiperfect ring. If $\{\epsilon_i\}_{i=1}^{m}$ and $\{f_j\}_{j=1}^{l}$ are complete sets of primitive orthogonal idempotents of $S$ then $m = l$, and there exist a unit $u$ of $S$ and a permutation $\sigma$ of $\{1, \ldots, m\}$ such that $u\epsilon_i u^{-1} = f_{\sigma(i)}$ for all $i = 1, \ldots, m$.*

Using this lemma, Fujita and Yoshimura provide the following criterion for isomorphic tiled orders:

**Theorem 3.2.6** (Fujita-Yoshimura, [13]). *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ and $\Gamma' = (\mathfrak{p}^{\mu'_{ij}})$ be two tiled orders. Then $\Gamma$ and $\Gamma'$ are isomorphic as rings if and only if there exists a diagonal matrix $d$ and a permutation matrix $w_\sigma$ such that $dw_\sigma \Gamma w_\sigma^{-1} d^{-1} = \Gamma'$.*

*Remark* 3.2.7. By the Skolem-Noether theorem, each isomorphism of orders $\psi : \Gamma \to \Gamma'$ corresponds to conjugation by an element $\xi \in GL_n(D)$. In the proof of Theorem 3.2.6, Fujita and Yoshimura make use of the following commutative diagram of isomorphisms

$$\begin{array}{ccc} \Gamma & \xrightarrow{\psi} & \Gamma' = \xi\Gamma\xi \\ \downarrow & & \downarrow \\ \xi_\sigma \Gamma \xi_\sigma^{-1} & \longrightarrow & u^{-1}\xi\Gamma\xi^{-1}u \end{array}$$

where $\xi_\sigma := dw_\sigma$ is the product of the diagonal and the permutation matrix from the statement of the theorem, and $u \in \Gamma'^{\times}$ is a unit from Lemma 3.2.5 such that

$\psi(e_{ii}) = ue_{\sigma^{-1}(i)\sigma^{-1}(i)}u^{-1}$, and show that $\Gamma' = \xi_\sigma \Gamma \xi_\sigma^{-1} = u^{-1}\xi\Gamma\xi^{-1}u$.

The idempotents $\{e_{ii}\}_{i=1}^n$ have the following trajectory under this commutative diagram

$$\begin{array}{ccc} e_{ii} & \overset{\psi}{\longmapsto} & ue_{\sigma^{-1}(i)\sigma^{-1}(i)}u^{-1} \\ \downarrow & & \downarrow \\ e_{\sigma^{-1}(i)\sigma^{-1}(i)} & \longmapsto & e_{\sigma^{-1}(i)\sigma^{-1}(i)} \end{array} .$$

**Corollary 3.2.8.** *Let $\mathcal{N}(\Gamma)$ be the normalizer of $\Gamma$ in $GL_n(D)$. Then the map*

$$\mathcal{N}(\Gamma) \to S_n$$
$$\xi \mapsto \sigma,$$

*where $\xi e_{ii}\xi^{-1} = ue_{\sigma^{-1}(i)\sigma^{-1}(i)}u^{-1}$ for some unit $u \in \Gamma^\times$, is a homomorphism.*

We have seen that conjugation by an element in $GL_n(D)$ induces an action on the building, since conjugating a maximal order corresponds to left-multiplying its corresponding homothety class. To investigate how the isomorphism in Theorem 3.2.6 acts on the distinguished vertices, we start with the following lemma:

**Lemma 3.2.9.** *Let $\Gamma$ and $\Gamma'$ be two tiled orders with polytopes in our fixed apartment $\mathcal{A}$ with distinguished vertices $[P_i]$ and $[P_i']$, and maximal orders corresponding to such vertices given by $\Lambda_i$ and $\Lambda_i'$. If the two tiled orders are conjugate by an element $\xi \in GL_n(D)$, then there exists $\sigma \in S_n$ such that $\xi e_{ii}\xi^{-1} = e_{\sigma^{-1}(i)\sigma^{-1}(i)}$ for all $i \leq n$, and we have an induced action on the vertices $\xi[P_i] = [P'_{\sigma^{-1}(i)}]$, and $\xi\Lambda_i\xi^{-1} = \Lambda'_{\sigma^{-1}(i)}$ for each $i \leq n$.*

*Proof.* Since the lattices $P_i$ are determined by the columns of $M_\Gamma$, we can identify $P_i$ with $\Gamma e_{ii}$. Then

$$\xi\Gamma e_{ii}\xi^{-1} = \xi\Gamma\xi^{-1}\xi e_{ii}\xi^{-1} = \Gamma' e_{\sigma^{-1}(i)\sigma^{-1}(i)}$$

which is identified with $[P'_{\sigma^{-1}(i)}]$. Therefore, $\xi[P_i] = [P'_{\sigma^{-1}(i)}]$, and $\xi \Lambda_i \xi^{-1} = \Lambda'_{\sigma^{-1}(i)}$.

□

Therefore, the maximal orders $\Lambda_i$ have the following trajectory under the commutative diagram from earlier:

$$\begin{array}{ccc} \Lambda_i & \xrightarrow{\psi} & \Lambda'_{\sigma^{-1}(i)} \\ \downarrow & & \downarrow \\ \Lambda'_{\sigma^{-1}(i)} & \longrightarrow & \Lambda'_{\sigma^{-1}(i)} \end{array}$$

since $u \in \Gamma'^{\times}$ implies $u \in (\Lambda'_{\sigma^{-1}(i)})^{\times}$ and $u\Lambda'_{\sigma^{-1}(i)}u^{-1} = \Lambda'_{\sigma^{-1}(i)}$.

We can now interpret Theorem 3.2.6 in terms of an action on the building for $SL_n(D)$.

**Corollary 3.2.10.** *Suppose $\Gamma$ and $\Gamma'$ are two tiled orders with polytopes $C_\Gamma$ and $C_{\Gamma'}$ in a fixed apartment $\mathcal{A}$, and $\psi : \Gamma \to \Gamma'$ an isomorphism. Then $\psi$ induces an action $\widetilde{\psi}$ on the building for $SL_n(D)$ such that $\widetilde{\psi}(C_\Gamma) = C_{\Gamma'}$. Moreover, there exists an isomorphism $\phi : \Gamma \to \Gamma'$ whose induced action $\widetilde{\phi}$ on the building gives an automorphism of the apartment $\mathcal{A}$ as a simplicial complex, and $\widetilde{\phi}|_{C_\Gamma} = \widetilde{\psi}|_{C_\Gamma}$.*

*Proof.* Since $\psi$ corresponds to conjugation by an element in $\xi \in GL_n(D)$, where $\xi\Gamma\xi^{-1} = \Gamma'$, $\psi$ induces a simplicial action on the building, where $\widetilde{\psi}([L]) = \xi[L]$. Clearly, conjugation by $\xi$ sends the set of maximal orders containing $\Gamma$ to the set of maximal orders containing $\Gamma'$. Since such maximal orders correspond to vertices on $\overline{C_\Gamma}$ and $\overline{C_{\Gamma'}}$, we have $\widetilde{\psi}(C_\Gamma) = C_{\Gamma'}$.

By Theorem 3.2.6 and the discussion following it, there exists a monomial matrix $\xi_\sigma$ such that conjugation by $\xi_\sigma$ gives an isomorphism $\phi : \Gamma \to \Gamma'$, where $u\phi(e_{ii})u^{-1} = \psi(e_{ii})$ for some unit $u \in \Gamma'^{\times}$. Note that monomial matrices preserve the apartment $\mathcal{A}$, so the induced action gives $\widetilde{\phi}(\mathcal{A}) = \mathcal{A}$. We are left to show that $\widetilde{\phi}|_{C_\Gamma} = \widetilde{\psi}|_{C_\Gamma}$. Since the distinguished vertices are extremal on $C_\Gamma$, uniquely determine $C_\Gamma$, and both $\widetilde{\psi}$ and $\widetilde{\psi}$ are simplicial maps, it suffices to show both maps agree on the distinguished vertices. However, this follows from Remark 3.2.7, Lemma 3.2.9 and the discussion following it, where $\xi\Lambda_i\xi^{-1} = \Lambda'_{\sigma^{-1}(i)} = \xi_\sigma\Lambda_i\xi_\sigma^{-1}$, for $\Lambda_i$ and $\Lambda'_j$ maximal orders corresponding to the distinguished vertices for $C_\Gamma$, and respectively, $C_{\Gamma'}$. □

Corollary 3.2.10 says that given two isomorphic tiled orders $\Gamma$ and $\Gamma'$ with polytopes in the same apartment $\mathcal{A}$, there exists an automorphism of $\mathcal{A}$ as a simplicial complex sending $C_\Gamma$ to $C_{\Gamma'}$ by rigid motions. Therefore, $C_\Gamma$ and $C_{\Gamma'}$ have the same "shap" and "size". We will call such polytopes *congruent*, a term that we will define more precisely in the following section.

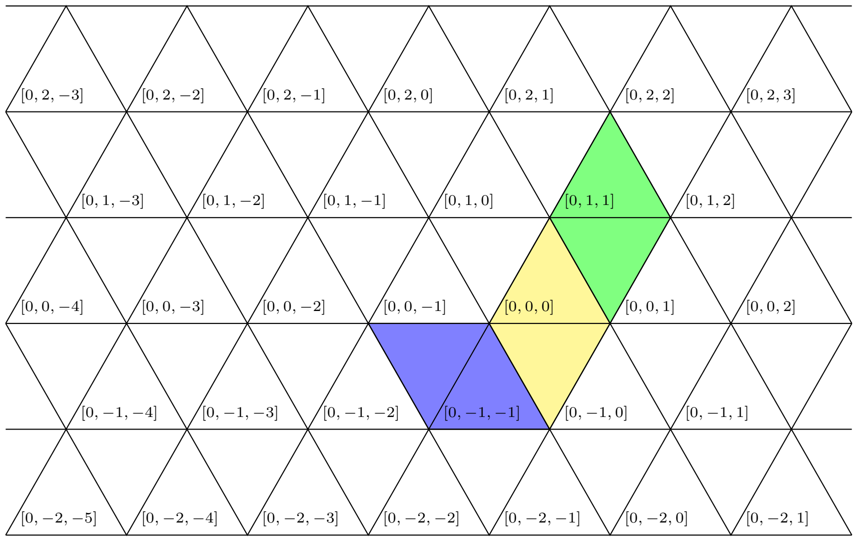
**Example 3.2.11.** Consider $\Gamma$ and $\Gamma'$ with exponent matrices

$$M_\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad M_{\Gamma'} = \begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}.$$

Then $\Gamma$ and $\Gamma'$ are isomorphic, and in particular $\xi\Gamma\xi^{-1} = \Gamma'$ for $\xi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{\pi} & 0 \\ \boldsymbol{\pi} & 0 & 0 \end{pmatrix}$. Notice that indeed $C_\Gamma$ depicted in Figure 3.3 in blue and $C_{\Gamma'}$ depicted in green are congruent in the usual sense of the word.

Moreover, $\xi$ decomposes as a product of a diagonal and a permutation matrix with $\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{\pi} & 0 \\ 0 & 0 & \boldsymbol{\pi} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The permutation matrix corresponds to the reflection with respect to the hyperplane $x_1 - x_3 = 0$, which gives the convex polytope $C_{\widetilde{\Gamma}}$ in yellow. The diagonal matrix then corresponds to translating $C_{\widetilde{\Gamma}}$ so that the vertex $[0, 1, 1]$ aligns with $[0, 2, 2]$, which then gives $C_{\Gamma'}$.

Figure 3.3:




Section 3.3

## Structural invariants

Consider a tiled order $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ with convex polytope $C_\Gamma$. To make notions of congruence precise, we need ways to get quantitative information about $C_\Gamma$. In unpublished work [47] (c.f. [33]), Zassenhaus introduced a set of *structural invariants* for tiled orders, defined by:

$$m_{ij\ell} = \mu_{ij} + \mu_{j\ell} - \mu_{i\ell}, \text{ for } 1 \leq i, j, \ell \leq n.$$

Note that because of the inequality in Equation (3.1), the structural invariants are nonnegative integers. In [40], where $n = 3$, these structural invariants encode the geometry of the convex polytope $C_\Gamma$. They correspond to side lengths of $C_\Gamma$ and distances between opposite bounding hyperplanes. In Example 3.1.3, we get

$m_{231} = 2, m_{321} = 1, m_{132} = 3, m_{312} = 1, m_{213} = 2$ are the five side lengths, $m_{123} = 0$ denotes a missing edge (that would have been on the hyperplane $x_1 - x_3 = 2$), and $m_{121} = 2, m_{131} = 4, m_{232} = 3$ are distances between the pairs of opposite hyperplanes $H_{12}, H_{21}$; $H_{13}, H_{31}$; and $H_{23}, H_{32}$.

In the general case $n \geq 3$, the $m_{ij\ell}$ still encode geometric data, which we restate in the following proposition.

**Proposition 3.3.1.** *For $i \neq j$, $m_{ij\ell}$ is the number of hyperplanes between the vertex $[P_\ell]$ and $H_{ij}$ (by definition, if $i = j$ then $m_{ij\ell} = 0$).*

*Proof.* Fix $i, j \leq n$. The vertex $[P_\ell]$ is on the hyperplane $x_i - x_j = \mu_{i\ell} - \mu_{j\ell}$. By Equation (3.1), $\mu_{i\ell} - \mu_{j\ell} \leq \mu_{ij}$. Thus, the number of hyperplanes between $H_{ij}$ (given by $x_i - x_j = \mu_{ij}$) and $[P_\ell]$ is $\mu_{ij} - (\mu_{i\ell} - \mu_{j\ell}) = \mu_{ij} - \mu_{i\ell} + \mu_{j\ell} = m_{ij\ell}$. □

The structural invariants detect when two distinguished vertices coincide, as showcased in the following corollaries.

**Corollary 3.3.2.** $m_{iji} = 0$ *if and only if* $[P_i] = [P_j]$.

*Proof.* Suppose $m_{iji} = 0$. We need to show there exists $t \in \mathbb{Z}$ such that $\mu_{\ell i} = \mu_{\ell j} + t$ for all $\ell \leq n$. Note that $m_{iji} = 0 \implies \mu_{ij} = -\mu_{ji}$, and by inequality (3.1) we also have $\mu_{\ell i} + \mu_{ij} \geq \mu_{\ell j}$. These conditions imply $\mu_{\ell i} - \mu_{ji} \geq \mu_{\ell j}$ and therefore $\mu_{\ell j} + \mu_{ji} \leq \mu_{\ell i}$. By inequality (3.1), this means $\mu_{\ell i} \leq \mu_{\ell j} + \mu_{ji} \leq \mu_{\ell i}$ so $\mu_{\ell j} + \mu_{ji} = \mu_{\ell i}$. Taking $t = \mu_{ji}$ proves our claim.

Now we prove the converse. Suppose $[P_i] = [P_j]$ for some $i, j \leq n$. Then

$$m_{iji} = \#\text{ of hyperplanes between } [P_i] \text{ and } H_{ij}.$$

At the same time,

$$m_{jij} = \# \text{ of hyperplanes between } [P_j] \text{ and } H_{ji}.$$

But $m_{iji} = m_{jij}$, and since $[P_i] = [P_j]$, it follows that

$$m_{iji} = \# \text{ of hyperplanes between } [P_i] \text{ and } H_{ji},$$

which is equal to 0 since $[P_i]$ is already on $H_{ji}$. □

**Corollary 3.3.3.** *$[P_i] = [P_j]$ if and only if $m_{rsi} = m_{rsj}$, $m_{ris} = m_{rjs}$ and $m_{irs} = m_{jrs}$ for all $r, s \leq n$.*

*Proof.* Suppose $[P_i] = [P_j]$. Then $m_{rsi} = m_{rsj}$ by Proposition 3.3.1. Now consider $m_{ris} - m_{rjs} = \mu_{ri} + \mu_{is} - \mu_{rj} - \mu_{js}$. Since $[P_i] = [P_j]$, in the proof of Corollary 3.3.2 we saw that $\mu_{ri} - \mu_{rj} = \mu_{ji} = -\mu_{ij}$. Plugging this into the difference above, we get two equalities

$$m_{ris} - m_{rjs} = -m_{ijs} = m_{jis} = 0,$$

since the structural invariants are nonnegative. Note that this means $m_{ijs} = 0 = m_{jis}$ for all $s \leq n$. Finally, another computation gives $m_{irs} - m_{jrs} = m_{ijs} - m_{ijr}$. By our previous remark, this is zero.

Now we prove the converse. Since $m_{rsi} = m_{rsj}$ for all $r, s \leq n$, this also applies if $r = i$ and $s = j$, so $m_{iji} = m_{ijj} = 0$, and by Corollary 3.3.2, $[P_i] = [P_j]$.

□

**Corollary 3.3.4.** *A tiled order $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ has full geometric rank if and only if all the $n$ homothety classes $[P_1], \ldots, [P_n]$ are distinct.*

*Proof.* $\Gamma$ has full geometric rank if and only if $\mu_{ij} + \mu_{ji} > 0$ for all $i, j$, if and only if $m_{iji} > 0$ for all $i, j \leq n$, and by Corollary 3.3.2, if and only if $[P_i] \neq [P_j]$ for all $i, j \leq n$. □

Since the structural invariants encode geometric data, we may use them to define properties of associated polytopes.

**Definition 3.3.5.** Let $\Gamma$ be a tiled order with structural invariants $m_{ij\ell}$, and $\Gamma'$ a tiled order with structural invariants $m'_{ij\ell}$. We say $C_\Gamma$ and $C_{\Gamma'}$ are **congruent** if there exists $\sigma \in S_n$ such that $m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq n$.

For practical purposes, since $m_{ijj} = 0$ and $m_{iji} = m_{ij\ell} + m_{ji\ell}$, we only need to compare relations on invariants of the form $m_{ij\ell}, i \neq j \neq \ell \neq i$ to determine whether two polytopes are congruent. In Example 3.2.11, $\Gamma$ has structural invariants

$$(m_{123}, m_{132}, m_{213}, m_{231}, m_{312}, m_{321}) = (1, 1, 0, 1, 0, 1),$$

and $\Gamma'$ has

$$(m'_{123}, m'_{132}, m'_{213}, m'_{231}, m'_{312}, m'_{321}) = (1, 0, 1, 0, 1, 1).$$

Then $m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for $\sigma = (13)$, and as we have already seen $C_\Gamma$ and $C_{\Gamma'}$ are congruent in the usual sense of the word. In addition, we have seen that $\Gamma$ and $\Gamma'$ are also isomorphic.

In fact, in [47] (see [33, II.6]), Zassenhaus shows that the structural invariants also encode the isomorphism class of the tiled order.

**Proposition 3.3.6** (Zassenhaus, [47])**.** *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ and $\Gamma' = (\mathfrak{p}^{\mu'_{ij}})$ be two tiled orders, and let $m_{ij\ell}$ and respectively, $m'_{ij\ell}$ be their structural invariants. Then $\Gamma$ and $\Gamma'$ are isomorphic if and only if there exists $\sigma \in S_n$ such that $m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $1 \leq i, j, \ell \leq n$.*

*Proof.* This is a part of Zassenhaus' result as described in [33, Prop. II.6]. Suppose the two orders are isomorphic. By Theorem 3.2.6, there exists a monomial matrix

$\xi \in GL_n(D)$ with $\Gamma' = \xi\Gamma\xi^{-1}$, where $\xi = (\boldsymbol{\pi}^{\alpha_i}\delta_{\sigma(i)j})$ for some $\sigma \in S_n$, $\alpha_i \in \mathbb{Z}$, and $\delta_{ij}$ the Kronecker delta. Let $\Gamma = (\mathfrak{p}^{\mu_{ij}}), \Gamma' = (\mathfrak{p}^{\mu'_{ij}})$. Conjugating by $\xi$ we deduce

$$\mu'_{ij} = \alpha_i - \alpha_j + \mu_{\sigma(i)\sigma(j)}.$$

Therefore,

$$\begin{aligned} m'_{ij\ell} &= \mu'_{ij} + \mu'_{j\ell} - \mu'_{i\ell} \\ &= \alpha_i - \alpha_j + \mu_{\sigma(i)\sigma(j)} + \alpha_j - \alpha_\ell + \mu_{\sigma(j)\sigma(\ell)} - \alpha_i + \alpha_\ell - \mu_{\sigma(i)\sigma(\ell)} \\ &= m_{\sigma(i)\sigma(j)\sigma(\ell)}. \end{aligned}$$

Conversely, suppose there is $\tau \in S_n$ such that $m'_{ij\ell} = m_{\tau(i)\tau(j)\tau(\ell)}$ for all $1 \leq i, j, \ell \leq n$. Then let $\alpha_i = \mu'_{i1} - \mu_{\tau(i)\tau(1)}$, $i \leq n$. Note that $\alpha_1 = 0$, and that also $\alpha_i = \mu_{\tau(1)\tau(i)} - \mu'_{1i}$, since

$$\mu'_{i1} + \mu'_{1i} = m'_{i1i} = m_{\tau(i)\tau(1)\tau(i)} = \mu_{\tau(i)\tau(1)} + \mu_{\tau(1)\tau(i)}.$$

If we let $\xi_{ij} = \boldsymbol{\pi}^{\alpha_i}\delta_{\tau(i)j}$, then the exponents of $\xi\Gamma\xi^{-1}$ are $\alpha_i - \alpha_j + \mu_{\tau(i)\tau(j)}$, which give

$$\begin{aligned} \alpha_i - \alpha_j + \mu_{\tau(i)\tau(j)} &= \mu'_{i1} - \mu_{\tau(i)\tau(1)} - \mu_{\tau(1)\tau(j)} + \mu'_{1j} + \mu_{\tau(i)\tau(j)} \\ &= \mu'_{i1} + \mu'_{1j} - \mu'_{ij} + \mu'_{ij} - \mu_{\tau(i)\tau(1)} - \mu_{\tau(1)\tau(j)} + \mu_{\tau(i)\tau(j)} \\ &= m'_{i1j} + \mu'_{ij} - m_{\tau(i)\tau(1)\tau(j)} \\ &= \mu'_{ij}, \end{aligned}$$

and therefore $\xi\Gamma\xi^{-1} = \Gamma'$, so the two orders are isomorphic.

□

**Corollary 3.3.7.** *Two tiled orders* $\Gamma$ *and* $\Gamma'$ *with polytopes* $C_\Gamma$ *and* $C_{\Gamma'}$ *in the same apartment are isomorphic if and only if* $C_\Gamma$ *and* $C_{\Gamma'}$ *are congruent.*

Note that in the proof of Proposition 3.3.6, if $m_{ij\ell} = m'_{ij\ell}$, then the resulting $\xi$ is a diagonal matrix. Since conjugation by a diagonal matrix translates vertices in the apartment, we have the following connection between the polytopes:

**Corollary 3.3.8.** *Let* $\Gamma = (\mathfrak{p}^{\mu_{ij}}), \Gamma' = (\mathfrak{p}^{\mu'_{ij}})$ *be two tiled orders, and let* $m_{ij\ell}$ *and* $m'_{ij\ell}$ *be their structural invariants. If* $m'_{ij\ell} = m_{ij\ell}$ *for all* $1 \leq i, j, \ell \leq n$*, then there exists a diagonal matrix acting on the apartment* $\mathcal{A}$ *by a translation, such that* $C_\Gamma$ *is translated to* $C_{\Gamma'}$*.*

Section 3.4

## Hereditary orders

Let $\Gamma \subseteq M_n(D)$ be a tiled order with exponent matrix $M_\Gamma = (\mu_{ij})$. Having developed the tools from the previous sections, we would like to interpret algebraic properties of tiled orders in terms of properties of their convex polytopes.

Recall Proposition 3.2.1, which described the complete set of isomorphism classes of injective and projective indecomposable $\Gamma$-lattices in $D^n$. These descriptions of $\Gamma$-lattices allow us to characterize the convex polytopes of some special kinds of orders. We start with *hereditary orders.*

**Definition 3.4.1.** We say an order $\Gamma$ is **left hereditary** if every left $\Gamma$-ideal is projective as a left $\Gamma$-module, and we say that an order $\Gamma$ is **strictly hereditary** if every left $\Gamma$-lattice is $\Gamma$-projective.

We can similarly define a **right hereditary** order, however in our case the two definitions coincide (see [20, Corollary 1.6.0]) and we will simply say **hereditary**. By Theorem 1.6 in [20], a tiled order $\Gamma$ is hereditary if and only if it is strictly hereditary.

In [16] and [17], Harada showed that every hereditary order in $M_n(D)$ is isomorphic to a tiled order whose exponent matrix is of the form

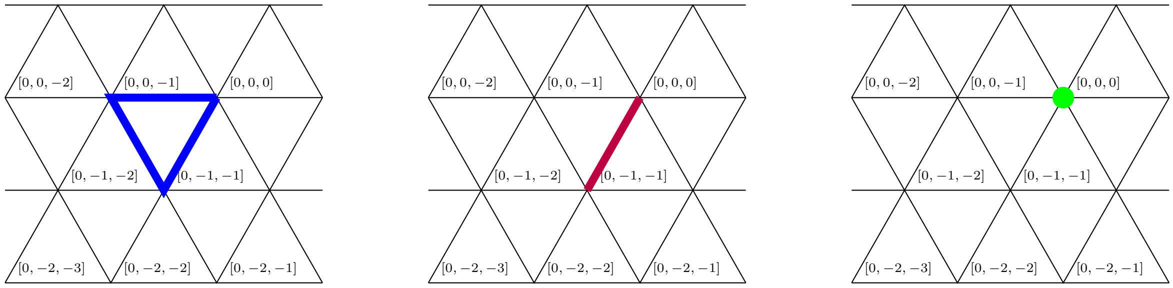
$$\left(\begin{array}{c:c:c:c} 0_{m_1\times m_1} & 1_{m_1\times m_2} & \cdots & 1_{m_1\times m_r} \\ \hdashline 0_{m_2\times m_1} & 0_{m_2\times m_2} & \cdots & 1_{m_2\times m_r} \\ \hdashline \vdots & \vdots & \cdots & \vdots \\ \hdashline 0_{m_r\times m_1} & 0_{m_r\times m_2} & \cdots & 0_{m_r\times m_r} \end{array}\right), \tag{3.2}$$

where $n = \sum_{i=1}^{r} m_i$ and $l_{i\times j}$ denotes the $i \times j$ matrix whose entries are all $l$. The question that arises is whether we can say anything about convex polytopes associated to hereditary orders. We start with an example.

**Example 3.4.2.** Every hereditary order in $M_3(D)$ is isomorphic to one of the tiled orders with the following exponent matrices:

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These are a complete set of isomorphism classes of tiled orders whose convex polytopes are simplices in the building. In the following figure, they correspond to the emphasized chamber, edge, and vertex.




**Proposition 3.4.3.** *Let $\Gamma \subseteq M_n(D)$ be an order. Then $\Gamma$ is hereditary if and only if it is a tiled order whose convex polytope is a simplex in the building for $SL_n(D)$.*

To prove the above proposition, we need the following rather technical lemma.

**Lemma 3.4.4.** *Suppose $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ is a tiled order with exponents $\mu_{ij} \in \{0, 1\}$, and that all the entries below the diagonal are zero. If any entry above the diagonal is zero, then all entries below it and to the left of it are zero as well. Furthermore, $\Gamma$ is of the form in Equation (3.2), and is therefore hereditary.*

*Proof.* We start with the first claim. Suppose there exists a zero entry $\mu_{ij} = 0$ for $i \leq j$, such that either an entry to the left or below it are 1. In the first case, this means there exists $\ell < j$ such that $\mu_{i\ell} = 1$. Since all the entries below the diagonal are 0, we get $m_{ij\ell} = \mu_{ij} + \mu_{j\ell} - \mu_{i\ell} = -1$, which contradicts that $\Gamma$ is an order. Similarly, if any of the entries below $\mu_{ij}$ are 1, there exists $l > i$ such that $\mu_{lj} = 1$, so $m_{lij} = \mu_{li} + \mu_{ij} - \mu_{lj} = -1$, which again contradicts that $\Gamma$ is an order. Note that this implies that all all entries in the largest rectangle whose upper right corner is $\mu_{ij}$ are zeros as well (so $\mu_{tl} = 0$ for all $t \geq i$ and $l \leq j$).

We proceed with the second claim, and start by dividing the exponent matrix into blocks. Let $m_1 \leq n$ be the smallest number such that $\mu_{1,m_1} = 0$, but the entry immediately to the right (if there is one) is 1. From the first paragraph, we conclude that all the entries in the first $m_1$ columns are zero, and all the entries in the first row to the right of $\mu_{1,m_1+1}$ are ones. Therefore, the exponent matrix has the form $\left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & & & \\ 0 & \cdots & 0 & & & \\ \hline \multicolumn{3}{c|}{0_{(n-m_1)\times m_1}} & & & \end{array}\right)$. If $\mu_{m_1,m_1+1} = 1$, by the first claim all the entries in the upper right square are ones. Suppose not, and $\mu_{m_1,m_1+1} = 0$. Then $\mu_{1m_1} + \mu_{m_1,m_1+1} - \mu_{1,m_1+1} = -1$, which contradicts that $\Gamma$ is an order. Therefore, the exponent matrix as the from $\left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ \hline \multicolumn{3}{c|}{0_{(n-m_1)\times m_1}} & & & \end{array}\right)$. We repeat the process letting $m_2 \leq n$ be the smallest such that $\mu_{m_1+1,m_1+m_2} = 0$, but the entry immediately to the right (if there is one) is 1. Then we get another diagonal block of zeros, and repeating until the end gives us a matrix of the form in Equation (3.2), which is hereditary per Harada's results. □

*Proof of Proposition 3.4.3.* Suppose $\Gamma$ is tiled, and $C_\Gamma$ is a simplex. Consider the set of structural invariants $\{m_{ij\ell}\}_{i,j,\ell \leq n}$. Since $C_\Gamma$ is a simplex, each $m_{ij\ell} \in \{0, 1\}$ by Proposition 3.3.1. From [33, Proposition II.6 (ii)], the tiled order $(\mathfrak{p}^{m_{ij1}})$ has the same structural invariants as $\Gamma$ and is therefore isomorphic to $\Gamma$, and we may assume that the order we started with is of this form.

So let $\Gamma = (\mathfrak{p}^{\mu_{ij}}) = (\mathfrak{p}^{m_{ij1}})$. Note that the exponents in the first column of $M_\Gamma$ are all 0, the remaining exponents are either 0 or 1, and that if $\mu_{ij} = 1$ then $\mu_{ji} = 0$. If all the exponents in $M_\Gamma$ below the diagonal are zero, then $\Gamma$ is hereditary by Lemma 3.4.4. If not, then we want to show $\Gamma$ is isomorphic to an order whose exponent matrix is upper triangular.

Let $\ell \leq n$ be the first column having an entry equal to 1 below the diagonal, and let $t > \ell$ be the smallest such that $\mu_{t\ell} = 1$. Then $\mu_{\ell t} = 0$ as remarked above. Let

$w_\tau$ be the permutation matrix associated to the transposition $\tau = (t\ell)$, and consider $\Gamma' = (\mathfrak{p}^{\mu'_{ij}}) = w_\tau \Gamma w_\tau^{-1}$. Then $\mu'_{ij} = \mu_{\tau(i)\tau(j)}$. We claim the following:

(1) $\mu'_{t\ell} = \mu_{\sigma(t)\sigma(\ell)} = \mu_{\ell t} = 0$.

(2) the conjugation does not change the first $\ell - 1$ columns, i.e. $\mu'_{ij} = \mu_{ij}$ for all $i \leq n$ and $j < \ell$. Note that $\mu'_{ij} = \mu_{\tau(i)\tau(j)} = \mu_{\tau(i)j}$ since $\tau$ fixes $j < \ell < t$. If $i \neq t, \ell$, the claim follows. If $i = t$ or $\ell$, then $\mu'_{ij} = 0 = \mu_{ij}$, since $j < \ell < t$ and all the entries under the diagonal in the first $\ell - 1$ columns are zero.

(3) the conjugation preserves the zeros in the $\ell$-th column between the diagonal entry and $\mu_{t\ell}$, i.e. $\mu'_{j\ell} = 0$ for $\ell < j < t$. First, $\mu'_{j\ell} = \mu_{\tau(j)\tau(\ell)} = \mu_{jt}$. If $\mu_{jt} = 1$, then $m_{j\ell t} = \mu_{j\ell} + \mu_{\ell t} - \mu_{jt} = -1$, which contradicts that $\Gamma$ is an order.

Therefore, conjugation by $w_\tau$ does not change the upper triangular structure of the entries in the first $\ell - 1$ columns, or those in the $\ell$-th column above the $(t, \ell)$ position, while at the same time changing the value in the $(t, \ell)$ position to 0. Iterating as necessarily, we get to a tiled order whose exponent matrix is upper triangular, and we are done with the forward direction.

Now suppose $\Gamma$ is hereditary, so it is isomorphic to a tiled order with exponent matrix $M_\Gamma$ described in Equation (3.2). Denote by $L_i$ the lattice $L_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0)$ for $0 \leq i \leq n - 1$. Then $L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_{n-1} \supsetneq \boldsymbol{\pi} L_0$ is a flag that defines a chamber in the apartment. Note that each column of $M_\Gamma$ corresponds to one of the lattices $L_i$, and that indeed any subset of such homothety classes of lattices gives us a simplex. $\square$

Let us see an example of the algorithm described in the proof of Proposition 3.4.3:

**Example 3.4.5.** Consider the tiled order $\Gamma$ has exponent matrix $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$. We have the flag $P_1 \supsetneq P_2 \supsetneq P_3 \supsetneq \boldsymbol{\pi} P_1$, and therefore $C_\Gamma$ is a 2-simplex.

Then $\ell = 2, t = 4$, $\tau = (24)$, and $w_\tau \Gamma w_\tau^{-1}$ has exponent matrix $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Now let $\ell = 3, t = 4$, we take $\tau = (34)$, and conjugating again gives the upper triangular exponent matrix $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

We can give an alternative interpretation of the proposition above. Recall that by Proposition 3.2.1, each isomorphism class of $\Gamma$-lattices in $D^n$ corresponds to a vertex on $\overline{C_\Gamma}$. Since $\Gamma$ is hereditary if and only if each $\Gamma$-lattice is projective, it follows that every vertex on $\overline{C_\Gamma}$ must be a distinguished vertex. Since there are at most $n$ distinguished vertices, and each distinguished vertex is an extremal vertex at the intersection of bounding hyperplanes for $C_\Gamma$, this can happen if and only if $C_\Gamma$ is a simplex.

Finally, we conclude with the following inclusion of orders in $M_n(D)$ and corresponding polytopes:

| | | | | |
|---|---|---|---|---|
| Maximal orders | $\subsetneq$ | Hereditary orders | $\subsetneq$ | Tiled orders |
| Vertices | $\subsetneq$ | Simplices | $\subsetneq$ | Convex polytopes |

Section 3.5

## The radical idealizer chain

Let's explore some of the implications of the properties of hereditary orders discussed in the previous section. For any order $\Lambda$ in a semisimple $k$-algebra, there is a canonical chain of orders, described in [2] and called the *radical idealizer chain* of $\Lambda$

$$\Lambda = \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_h = \Lambda_{h+1},$$

terminating in a hereditary order $\Lambda_h$ called the *head order* of $\Lambda$, and each $\Lambda_{i+1}$ is the *idealizer* of the *Jacobson radical* of $\Lambda_i$. In this section, we define all the terms above, and investigate chains for tiled orders.

**Definition 3.5.1.** The Jacobson radical of a ring $A$, denoted by $J(A)$, is the intersection of all maximal left ideals of $A$.

One can easily find the Jacobson radical of a tiled order:

**Lemma 3.5.2** (Plesken, [33])**.** *Let* $\Gamma = (\mathfrak{p}^{\mu_{ij}}) \subseteq M_n(D)$ *be a tiled order with distinguished vertices* $[P_\ell]$*. Then* $J(\Gamma) = (\mathfrak{p}^{\widetilde{\mu}_{ij}})$*, where*

$$\widetilde{\mu}_{ij} = \mu_{ij} + \tilde{\delta}_{ij},$$

*and* $\tilde{\delta}_{ij} = 1$ *if* $[P_i] = [P_j]$*.*

Note that $\tilde{\delta}_{ij}$ is not the regular Kronecker delta; in particular, if $C_\Gamma$ does not have full geometric rank, there exist $i \neq j$ such that $[P_i] = [P_j]$ by Corollary 3.3.4. By definition, $J(\Gamma)$ is an ideal in $\Gamma$, and therefore a $\Gamma$-lattice. Moreover,

**Lemma 3.5.3.** *Each of the columns in* $M_{J(\Gamma)}$ *corresponds to a vertex on* $\overline{C_\Gamma}$*.*

*Proof.* Consider the $\ell$-th column of $M_{J(\Gamma)}$, with entries

$$[\widetilde{\mu}_{1\ell}, \widetilde{\mu}_{2\ell}, \ldots, \widetilde{\mu}_{n\ell}].$$

To show that such a vertex is on $\overline{C_\Gamma}$, we need $\widetilde{\mu}_{i\ell} - \widetilde{\mu}_{j\ell} \leq \mu_{ij}$. But

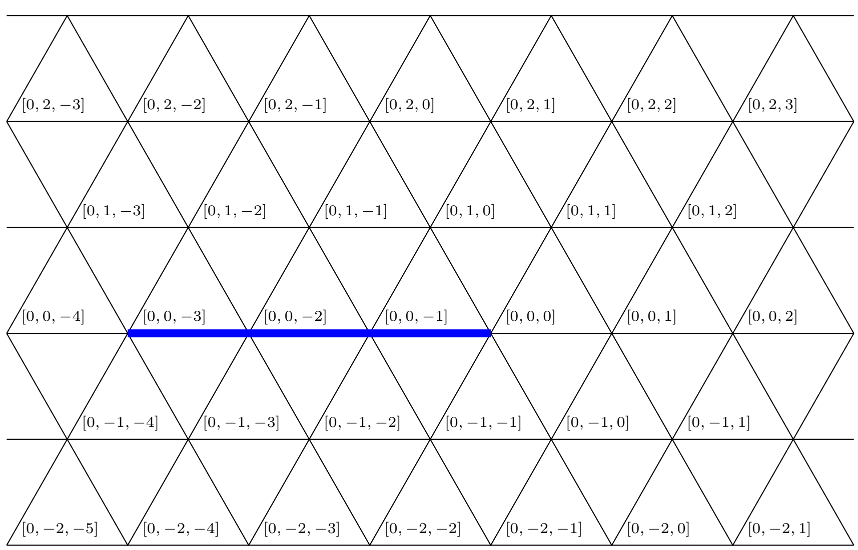
$$\mu_{ij} - \widetilde{\mu}_{i\ell} + \widetilde{\mu}_{j\ell} = \mu_{ij} - \mu_{i\ell} - \widetilde{\delta}_{i\ell} + \mu_{j\ell} + \widetilde{\delta}_{j\ell} = m_{ij\ell} - \widetilde{\delta}_{i\ell} + \widetilde{\delta}_{j\ell}.$$

Since $m_{ij\ell} \geq 0$, the only case the above expression would be negative is if $m_{ij\ell} = 0$, $\widetilde{\delta}_{i\ell} = 1$ and $\widetilde{\delta}_{j\ell} = 0$. The latter two conditions mean $[P_i] = [P_\ell] \neq [P_j]$. By Proposition 3.3.1, if $[P_i] = [P_\ell]$ then $m_{ij\ell} = m_{iji}$, so we need to check when $m_{iji} = 0$. From Corollary 3.3.2 this only happens if $[P_i] = [P_j]$, which is a contradiction.

□

**Example 3.5.4.** Consider $\Gamma$ with $M_\Gamma = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$. In Figure 3.4, $C_\Gamma$ is the emphasized 1-complex. Since $[P_1] = [P_2]$, $J(\Gamma)$ has exponent matrix $M_{J(\Gamma)} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$. Note that the columns of $M_{J(\Gamma)}$ indeed correspond to vertices on $\overline{C_\Gamma}$.

Figure 3.4:



**Definition 3.5.5** (Reiner, p.109 of [34])**.** Let $R$ be a noetherian integral domain with quotient field $F$, and $A$ a finite-dimensional $F$-algebra. Let $M$ be any full $R$-lattice in $A$. The left order of $M$ is defined as

$$O_L(M) = \{x \in A \,:\, xM \subseteq M\}.$$

The right order of $M$ is defined as

$$O_R(M) = \{x \in A \,:\, Mx \subseteq M\}.$$

We would like to characterize the left and right orders of $J(\Gamma)$.

**Proposition 3.5.6.** *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a tiled order with Jacobson radical $J(\Gamma)$ and exponent matrix $M_{J(\Gamma)} = (\widetilde{\mu}_{ij})$. Then $O_L(J(\Gamma))$ is the tiled order whose convex polytope $C_{O_L(J(\Gamma))}$ is the convex hull of the vertices that correspond to the columns of $M_{J(\Gamma)}$. Moreover, $O_L(J(\Gamma)) = O_R(J(\Gamma))$.*

*Proof.* Let $(\mathfrak{p}^{b_{ij}})$ be the tiled order whose convex polytope is given by the convex hull of the vertices that correspond to the columns of $M_{J(\Gamma)}$, which we encode by $[L_\ell] = [\widetilde{\mu}_{1\ell}, \dots, \widetilde{\mu}_{n\ell}]$. Then the $b_{ij}$ are the smallest values that satisfy the relations $\widetilde{\mu}_{i\ell} - \widetilde{\mu}_{j\ell} \leq b_{ij}$ for all $\ell \leq n$, so $b_{ij} = \max_\ell(\widetilde{\mu}_{i\ell} - \widetilde{\mu}_{j\ell})$.

The next step is to show that $O_L(J(\Gamma))$ is tiled; it suffices to show that it contains the set of primitive orthogonal idempotents $e_{ii}$, where $e_{ij}$ is the $n \times n$ matrix with all zeros, except for 1 in the $(i,j)$-position. Indeed,

$$e_{ii}J(\Gamma) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \mathfrak{p}^{\widetilde{\mu}_{i1}} & \mathfrak{p}^{\widetilde{\mu}_{i2}} & \cdots & \mathfrak{p}^{\widetilde{\mu}_{in}} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \subseteq J(\Gamma).$$

Since $O_L(J(\Gamma))$ is tiled, we have $O_L(J(\Gamma)) = (\mathfrak{p}^{a_{ij}})$ for some $(a_{ij}) \in \mathbb{Z}_{n\times n}$. In particular, $\boldsymbol{\pi}^{a_{ij}}e_{ij} \in O_L(J(\Gamma))$ for all $i,j \leq n$. Since $\boldsymbol{\pi}^{\widetilde{\mu}_{j\ell}}e_{j\ell} \in J(\Gamma)$ for all $j,\ell \leq n$, we must have $\boldsymbol{\pi}^{a_{ij}}e_{ij}\boldsymbol{\pi}^{\widetilde{\mu}_{j\ell}}e_{j\ell} \in J(\Gamma)$ for all $i,j,\ell \leq n$, in which case

$$\boldsymbol{\pi}^{a_{ij}+\widetilde{\mu}_{j\ell}} \in \mathfrak{p}^{\widetilde{\mu}_{i\ell}} \implies a_{ij} + \widetilde{\mu}_{j\ell} \geq \widetilde{\mu}_{i\ell} \quad \text{for all} \quad i,j,\ell \leq n.$$

But this means $a_{ij} \geq \widetilde{\mu}_{i\ell} - \widetilde{\mu}_{j\ell}$ for all $i,j,\ell \leq n$. The smallest such value for $a_{ij}$ is $\max_\ell(\widetilde{\mu}_{i\ell} - \widetilde{\mu}_{j\ell}) = b_{ij}$, so $(\mathfrak{p}^{a_{ij}}) = (\mathfrak{p}^{b_{ij}})$, and we are done with the first claim. The second claim follows from analogous arguments. □

The *radical idealizer* of an order $\Gamma$ in a semisimple $k$-algebra is defined as (see [2])

$$Id(J(\Gamma)) = O_L(J(\Gamma)) \cap O_R(J(\Gamma)).$$

Since for tiled orders $O_L(J(\Gamma)) = O_R(J(\Gamma))$, the radical idealizer of $\Gamma$ is

$$Id(J(\Gamma)) = O_L(J(\Gamma)) = O_R(J(\Gamma)).$$

When $\Gamma$ is hereditary, we have the following result, proven in theorems (39.11) and (39.14) of [34].

**Theorem 3.5.7** (Reiner, [34]). *Let $k$ be a non-archimedean local field with valuation ring $R$, $\Gamma$ an $R$-order in the central simple $k$-algebra $B$, where $B \cong M_n(D)$, and $D$ is a central division $k$-algebra. Then $\Gamma$ is hereditary if and only if $Id(J(\Gamma)) = \Gamma$.*

Now consider $\Gamma$ a tiled order and the chain

$$\Gamma = \Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots \subsetneq \Gamma_h = \Gamma_{h+1}$$

introduced at the beginning of the section. We construct the chain by iterating the process of taking the radical idealizer and get $\Gamma_2 = Id(J(\Gamma_1))$, $\Gamma_3 = Id(J(\Gamma_2))$ etc. By Lemma 3.5.3, each column of $M_{J(\Gamma)}$ gives an internal vertex of the polytope $C_\Gamma$, and by Proposition 3.5.6 $C_{Id(J(\Gamma))}$ is the convex hull of these vertices. Since $\Gamma_i \subseteq \Gamma_{i+1}$, the polytopes at each step of the chain become smaller. Theorem 3.5.7 says that the chain terminates only once we hit a simplex, in which case the columns of $M_{J(\Gamma_h)}$ are the permuted distinguished vertices of $C_{\Gamma_h}$.

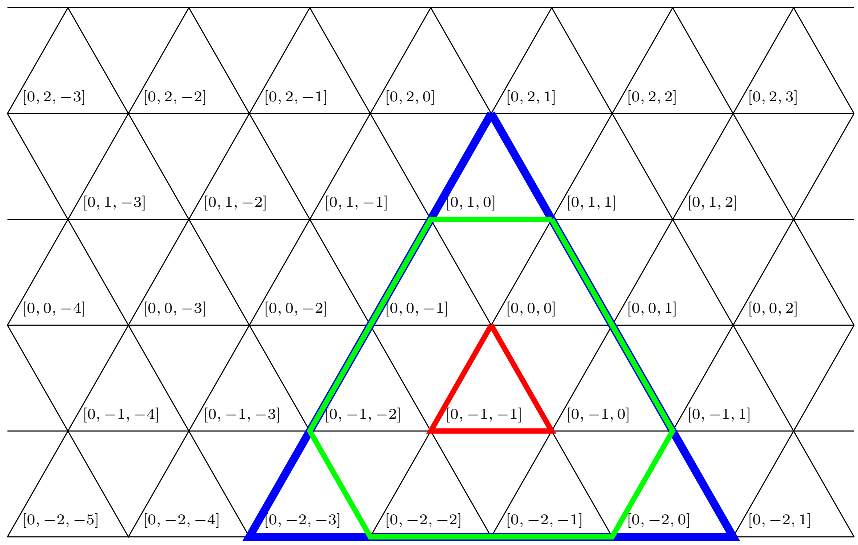
**Example 3.5.8.** Consider $\Gamma_1$ with exponent matrix $M_{\Gamma_1} = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}$, and polytope outlined in blue in Figure 3.5. Then

$$M_{J(\Gamma_1)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \quad \Longrightarrow \quad M_{\Gamma_2} = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix},$$

with convex polytope outlined in green.

Reiterating, we get $\Gamma_3 = \Gamma_h$, where $M_{\Gamma_3} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, with convex polytope outlined in red.

Figure 3.5:




Section 3.6

## Gorenstein tiled orders

In this subsection, we investigate the dual nature of the vertices introduced in Remark 3.2.4.

**Definition 3.6.1.** Let $L$ be an $\Delta$-lattice. The dual lattice $L^*$ is given by

$$L^* = \mathrm{Hom}_\Delta(L, \Delta).$$

In [48], duals of irreducible $\Gamma$-lattices are computed; the right irreducible $\Gamma$-lattice

$L = \begin{pmatrix}\mathfrak{p}^{m_1} & \dots & \mathfrak{p}^{m_n}\end{pmatrix}$ has irreducible dual left $\Gamma$-lattice given by

$$L^* = \begin{pmatrix}\mathfrak{p}^{-m_1} \\ \vdots \\ \mathfrak{p}^{-m_n}\end{pmatrix}. \tag{3.3}$$

If we consider $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ as a right module over itself, then $\Gamma \cong \bigoplus_{i=1}^{n} \begin{pmatrix}\mathfrak{p}^{\mu_{i1}} & \dots & \mathfrak{p}^{\mu_{in}}\end{pmatrix}$ is isomorphic to the direct sum of the lattices corresponding to the rows of $\Gamma$. Therefore

$$\Gamma^* = \mathrm{Hom}_\Delta(\Gamma_\Gamma, \Delta) \cong \bigoplus_{i=1}^{n} \begin{pmatrix}\mathfrak{p}^{-\mu_{i1}} \\ \mathfrak{p}^{-\mu_{i2}} \\ \vdots \\ \mathfrak{p}^{-\mu_{in}}\end{pmatrix}$$

as a left $\Gamma$-module.

**Definition 3.6.2** (Roggenkamp et al, [35])**.** A tiled order $\Gamma$ is called a **Gorenstein tiled order** if $\Gamma^*$ is a projective left $\Gamma$-lattice.

As a direct sum of indecomposable $\Gamma$-lattices, $\Gamma^*$ is a projective $\Gamma$-lattice if and only if each of the direct summands is projective. From Proposition 3.2.1 we have a complete description of the isomorphism classes of projective indecomposable $\Gamma$-lattices in $D^n$. In particular, from Equation (3.3) and Proposition 3.2.1, we conclude that $\Gamma$ is a Gorenstein tiled order if and only if the set of distinguished vertices $[P_i]$ coincides with the set of vertices given by $[R_j]$ (see Proposition 3.2.1).

**Example 3.6.3.** Let $\Gamma$ have exponent matrix

$$M_\Gamma = \begin{pmatrix}0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0\end{pmatrix}.$$

Taking $P_i$ and $R_j$ as defined in Proposition 3.2.1,

$$R_1 \cong P_3 \qquad R_2 \cong P_1 \qquad R_3 \cong P_4 \qquad R_4 \cong P_2.$$

When all the distinguished vertices are distinct (so $m_{iji} \neq 0$ when $i \neq j$) and $\Gamma$ is a Gorenstein tiled order, we get a permutation of the vertices as stated in [35, Theorem 1.4]:

**Theorem 3.6.4** (Roggenkamp et al, [35])**.** *A tiled order $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ with full geometric rank is a Gorenstein tiled order if and only if there exists a permutation $\sigma \in S_n$, called a **Kirichenko permutation**, such that*

$$\mu_{ij} + \mu_{j\sigma(i)} = \mu_{i\sigma(i)} \quad \textit{for all} \quad i, j \leq n.$$

In Example 3.6.3, the permutation is $\sigma = (1342)$.

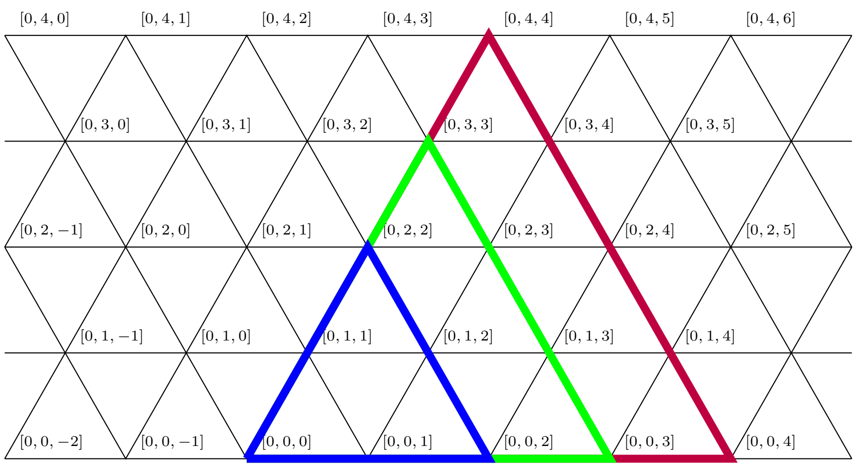
One type of polytopes whose corresponding tiled orders are Gorenstein are simplices; indeed, computing the dual of the general hereditary order from Equation (3.2) allows us to conclude that hereditary orders are Gorenstein tiled orders, and we have the following inclusion:

$$\text{Maximal orders} \subsetneq \text{Hereditary orders} \subsetneq \text{Gorenstein tiled orders}$$

At the same time, any **scaling** of a hereditary order is a Gorenstein tiled order, where the scaling of an order $\Gamma$ by $s \in \mathbb{Z}_{>0}$ is the tiled order with exponent matrix $sM_\Gamma$.

**Example 3.6.5.** Consider Gorenstein tiled orders in $M_3(D)$. Any scaling of a chamber is a Gorenstein tiled order. In fact, this is the only type of Gorenstein tiled orders in $M_3(D)$ with full geometric rank. For $n \geq 4$, however, there are Gorenstein tiled orders with full geometric rank that are not scalings of chambers. One such example is the tiled order in Example 3.6.3.

Figure 3.6: Scalings of chambers correspond to Gorenstein tiled orders.




Section 3.7

## Normalizers of tiled orders as symmetries of $C_\Gamma$

Let $\Gamma \subseteq M_n(D)$ be a tiled order with exponent matrix $M_\Gamma = (\mu_{ij})$, and structural invariants $\{m_{ij\ell} = \mu_{ij} + \mu_{j\ell} - \mu_{i\ell} : i, j, \ell \leq n\}$. In this section, we first interpret the *normalizer* $\mathcal{N}(\Gamma) = \{\xi \in GL_n(D) : \xi\Gamma\xi^{-1} = \Gamma\}$ of a tiled order in terms of symmetries of its associated convex polytope $C_\Gamma$.

We start with normalizers of maximal orders. Consider a maximal order $\Lambda$ in $M_n(k)$ for $k$ a non-archimedean local field with valuation ring $R$. Since each maximal order is conjugate to $M_n(R)$ by [34, (17.3)], it follows from [34, (37.26)] that $\mathcal{N}(\Lambda) =$

$k^\times\Lambda^\times$. In the case where our algebra $M_n(D)$ is of matrices over a local division ring, the result is analogous, but we need to be more careful.

Let $D$ be a central division algebra of degree $m$ over the non-archimedean local field $k$, with unique maximal $R$-order $\Delta$ and uniformizer $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}^m = \pi$ (see Section 14 in [34]).

Let $\Lambda = M_n(\Delta)$, then $\Lambda^\times = GL_n(\Delta)$.

**Proposition 3.7.1.** *For $\Lambda = M_n(\Delta)$, the normalizer is given by $\mathcal{N}(\Lambda) = D^\times\Lambda^\times$.*

*Proof.* We follow a similar approach to the discussion in [25, Section 3.1]. Since $\Delta$ is the unique maximal order in $D$, note that $x\Delta x^{-1} = \Delta$ for any $x \in D^\times$. Embedding $D^\times \hookrightarrow GL_n(D)$ diagonally, $xM_n(\Delta)x^{-1} = M_n(x\Delta x^{-1}) = M_n(\Delta)$, so $D^\times \subseteq \mathcal{N}(\Lambda)$. But clearly $\Lambda^\times \subseteq \mathcal{N}(\Lambda)$, and therefore $D^\times\Lambda^\times \subseteq \mathcal{N}(\Lambda)$.

Now we prove the other containment. From (37.25)-(37.27) of [34],

$$\mathcal{N}(\Lambda)/k^\times\Lambda^\times \cong \mathbb{Z}/m\mathbb{Z}. \tag{3.4}$$

By [34, 17.3], $\boldsymbol{\pi}\Lambda$ is the unique two-sided ideal of $\Lambda$ which implies $\boldsymbol{\pi} \in \mathcal{N}(\Lambda)$. Since $m$ is the smallest power such that $\boldsymbol{\pi}^m \in k^\times$, it follows that the normalizer is generated by the set $\{\boldsymbol{\pi}, k^\times\Lambda^\times\}$. We already have $D^\times\Lambda^\times \subseteq \mathcal{N}(\Lambda)$, so $D^\times\Lambda^\times \subseteq \langle\boldsymbol{\pi}, k^\times\Lambda^\times\rangle$. On the other hand, $\langle\boldsymbol{\pi}, k^\times\Lambda^\times\rangle \subseteq D^\times\Lambda^\times$ since $\boldsymbol{\pi} \in D^\times$. Thus $\mathcal{N}(\Lambda) = D^\times\Lambda^\times$. ☐

Next, we need more information about normalizers of other maximal orders. By [34, (17.3)], all maximal orders in $M_n(D)$ are conjugate to $M_n(\Delta)$, and we have the following easy corollary:

**Corollary 3.7.2.** *Let $\xi \in GL_n(D)$. Then*

$$\mathcal{N}(\xi M_n(\Delta)\xi^{-1}) = \xi\mathcal{N}(M_n(\Delta))\xi^{-1} = \xi D^\times GL_n(\Delta)\xi^{-1}.$$

For some maximal orders, the normalizer has a simplified form.

**Lemma 3.7.3.** *Consider an apartment containing the vertex $[0, 0, \ldots, 0]$ corresponding to $M_n(\Delta)$, and let $\Lambda$ be a maximal order whose vertex lies in this apartment. Then $\mathcal{N}(\Lambda) = D^\times\Lambda^\times$.*

*Proof.* Every maximal order $\Lambda$ in this apartment is of the form $\Lambda = \xi M_n(\Delta)\xi^{-1}$ for some diagonal matrix $\xi \in GL_n(D)$. By Corollary 3.7.2,

$$\mathcal{N}(\Lambda) = \xi D^\times GL_n(\Delta)\xi^{-1}.$$

We need to show $\xi D^\times \mathrm{GL}_n(\Delta)\xi^{-1} = D^\times\xi\mathrm{GL}_n(\Delta)\xi^{-1} = D^\times\Lambda^\times$.

Let $d \in D^\times$, and $\xi$ the diagonal matrix above. Then we can find integers $m, m_i \in \mathbb{Z}$ and units $u, u_i \in \Delta^\times$ for $i \leq n$ such that $d = \boldsymbol{\pi}^m u$ and $\xi = (\boldsymbol{\pi}^{m_i} u_i \delta_{ij})$, where $\delta_{ij}$ is the Kronecker delta. For any $a \in D$, there are unique units $a', a'' \in \Delta^\times$ and integer $k \in \mathbb{Z}$ such that $a = \boldsymbol{\pi}^k a' = a''\boldsymbol{\pi}^k$, where $a'$ and $a''$ are not necessarily equal (see [34, page 139]). A computation shows there exist units $v_i \in \Delta^\times$ such that

$$\xi d = (\boldsymbol{\pi}^{m_i} u_i \delta_{ij} \boldsymbol{\pi}^m u)_{ij} = (\boldsymbol{\pi}^m u \boldsymbol{\pi}^{m_i} u_i \delta_{ij} v_i)_{ij} = d\xi(v_i\delta_{ij})_{ij},$$

where consequently $(v_i\delta_{ij})_{ij} \in GL_n(\Delta)$.

Thus, $\xi D^\times \subseteq D^\times \xi GL_n(\Delta)$. Similarly, $D^\times \xi \subseteq \xi D^\times GL_n(\Delta)$, and therefore

$$\xi D^\times \mathrm{GL}_n(\Delta)\xi^{-1} = D^\times \xi \mathrm{GL}_n(\Delta)\xi^{-1}.$$

□

The above description of normalizers of maximal orders contains information to characterize normalizers of tiled orders.

**Proposition 3.7.4** (Shemanske, [40]). *Let $\Gamma$ be a tiled order whose convex polytope lies in the apartment containing the vertex $[0, 0, \ldots, 0]$ corresponding to $M_n(\Delta)$. Let $t$ be the number of distinct distinguished vertices of $C_\Gamma$. Then there is a homomorphism $\phi : \mathcal{N}(\Gamma) \to S_t$ with* $\ker(\phi) = D^\times \Gamma^\times$.

*Proof.* We use the ideas in Section 3 of [40]. Let $\{\Lambda_1, \ldots, \Lambda_t\}$ be the maximal orders containing $\Gamma$ corresponding to the $t$ distinguished vertices, so $\Gamma = \bigcap_{i=1}^t \Lambda_t$. Let $\xi \in \mathcal{N}(\Gamma)$, then conjugation of $\Gamma$ by $\xi$ gives an automorphism of $\Gamma$. Similar to the proof of Lemma 3.2.10, $\xi$ permutes the distinguished vertices $[P_i]$ of $C_\Gamma$. This gives an action of $\mathcal{N}(\Gamma)$ on the set of $t$ distinguished vertices, and therefore a homomorphism $\phi : \mathcal{N}(\Gamma) \to S_t$. Next we show that the kernel $\ker(\phi) = D^\times \Gamma^\times$. Since $\Gamma^\times = \cap_{i=1}^t \Lambda_i^\times$, conjugation by an element in $\Gamma^\times$ will fix each of these maximal orders and therefore each distinguished vertex. Since $x \in D^\times I_n$ gives $x\Gamma x^{-1} = \cap_{i=1}^t x\Lambda_i x^{-1} = \cap_{i=1}^t \Lambda_i = \Gamma$, it follows that $D^\times \Gamma^\times \subseteq \ker \phi$.

On the other hand, if $\xi$ fixes each distinguished vertex, then $\xi$ normalizes each maximal order $\Lambda_1, \Lambda_2, \ldots, \Lambda_t$ corresponding to each distinguished vertex, so $\xi \in \cap_{i=1}^t \mathcal{N}(\Lambda_t) = \cap_{i=1}^t D^\times \Lambda_i^\times$, the latter equality from Lemma 3.7.3. We claim that $\cap_{i=1}^t D^\times \Lambda_i^\times = D^\times \cap_{i=1}^t \Lambda_i^\times = D^\times \Gamma^\times$.

We proceed to prove the first equality. Clearly $D^\times \cap_{i=1}^t \Lambda_i^\times \subseteq \cap_{i=1}^t D^\times \Lambda_i^\times$. To show the nontrivial containment, suppose $\xi \in \cap_{i=1}^{t} D^\times \Lambda_i^\times$. Then we can write $\xi = \boldsymbol{\pi}^{a_1}\lambda_1 = \boldsymbol{\pi}^{a_2}\lambda_2 = \cdots = \boldsymbol{\pi}^{a_t}\lambda_t$, where each $\lambda_i \in \Lambda_i^\times$. Taking the reduced norm, we get $\mathrm{nr}_{M_n(D)/k}(\boldsymbol{\pi}) = (-1)^{m-1}\pi \in k$ and $\mathrm{nr}_{M_n(D)/k}(\lambda_i) \in R^\times$ for all $i \leq t$. Then $\mathrm{nr}_{M_n(D)/k}(\xi) = \pi^{na_1}u_1 = \pi^{na_2}u_2 = \cdots = \pi^{na_t}u_t$ for units $u_i \in R^\times$, so $a_1 = a_2 = \cdots = a_t =: a$ and $\xi = \boldsymbol{\pi}^a\lambda$ for some $\lambda \in \cap_{i=1}^{t}\Lambda_i^\times$. Therefore, $\cap_{i=1}^{t} D^\times\Lambda_i^\times \subseteq D^\times \cap_{i=1}^{t} \Lambda_i^\times$ and the proposition holds. □

*Remark* 3.7.5. If $\Gamma$ is a tiled order whose convex polytope lies in a different apartment, then $\Gamma = \zeta\Gamma'\zeta^{-1}$ for some $\zeta \in GL_n(D)$, and we get a homomorphism $\mathcal{N}(\Gamma) = \zeta\mathcal{N}(\Gamma')\zeta^{-1} \to S_t$ with kernel $\zeta D^\times \Gamma'^\times \zeta^{-1}$.

Suppose $\xi \in \mathcal{N}(\Gamma)$. Analogous to Corollary 3.2.10 and the discussion following it, $\xi$ permutes the distinguished vertices of $C_\Gamma$ by rigid motions, and Proposition 3.7.4 allows us to think of elements in the normalizer as inducing a "symmetry" on $C_\Gamma$. We will therefore refer to elements of $\mathcal{N}(\Gamma)/D^\times\Gamma^\times$ as the "symmetries of $C_\Gamma$", and we associate to $\xi \in \mathcal{N}(\Gamma)/D^\times\Gamma^\times$ the permutation $\sigma_\xi := \phi(\xi)$.

**Example 3.7.6.** For $n = 3$ and $\Gamma$ with full geometric rank, the symmetries of $C_\Gamma$ are given by a subgroup of $S_3$ as illustrated below.

Let $\Gamma_1$ be the tiled order with exponent matrx $M_{\Gamma_1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix}$ and polytope in Figure 3.7. We see that the symmetries correspond to a folding, interchanging $[P_1]$ and $[P_3]$, and fixing $[P_2]$. Let $\xi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{\pi}^2 & 0 \\ \boldsymbol{\pi}^4 & 0 & 0 \end{pmatrix}$. Then $\xi \in \mathcal{N}(\Gamma_1)$ with $\phi(\xi) = (1\,3)$, and $\mathcal{N}(\Gamma_1)/D^\times{\Gamma_1}^\times \cong \mathbb{Z}/2\mathbb{Z}$.

Let $\Gamma_2$ with exponent matrix $M_{\Gamma_2} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 3 & 3 & 0 \end{pmatrix}$ and polytope in Figure 3.8. We see that any two distinguished vertices can be interchanged. Let $\xi_1 =$

Figure 3.7:

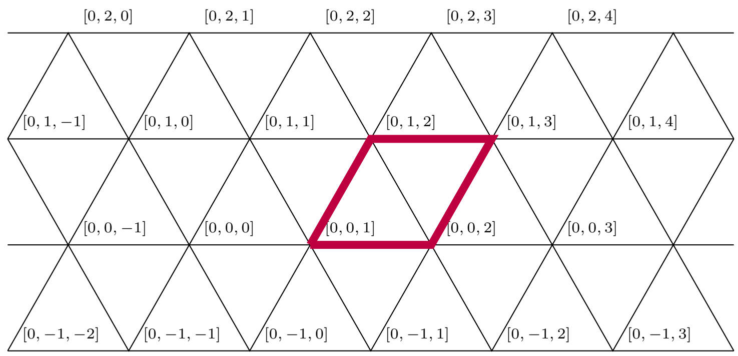
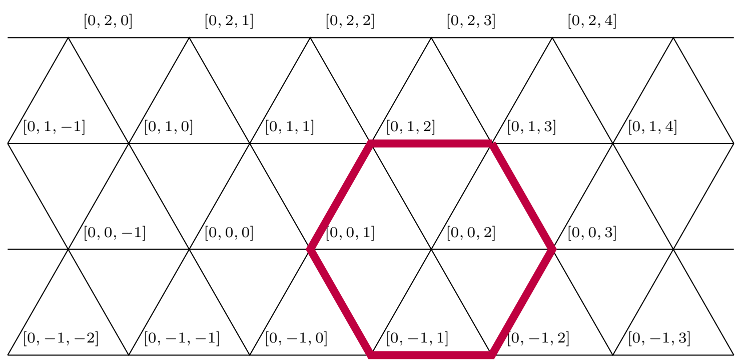



Figure 3.8:

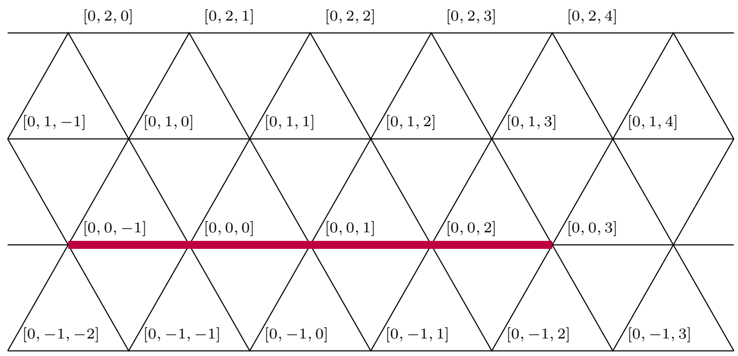



$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \boldsymbol{\pi}^2 & 0 \\ \boldsymbol{\pi}^4 & 0 & 0 \end{pmatrix}$, and $\xi_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\xi_1, \xi_2 \in \mathcal{N}(\Gamma_2)$ with $\phi(\xi_1) = (1\,3)$ and $\phi(\xi_2) = (12)$. Therefore $\mathcal{N}(\Gamma_2)/D^\times\Gamma_2^\times \cong S_3$.

Now consider the tiled order $\Gamma_3$ with exponent matrix $M_{\Gamma_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 3 & 3 & 0 \end{pmatrix}$ and polytope depicted in Figure 3.9.

Since $[P_1] = [P_2] \neq [P_3]$, we have two distinguished vertices, so the symmetries of $C_{\Gamma_3}$ is a subgroup of $S_2$. Suppose there exists $\xi \in GL_n(D)$ such that $\xi[P_1] = [P_3]$ and $\xi[P_3] = [P_1]$. Representing the type of a vertex $[L]$ by $t(L)$, Since $t(P_1) \equiv 0 \pmod 3$

Figure 3.9:



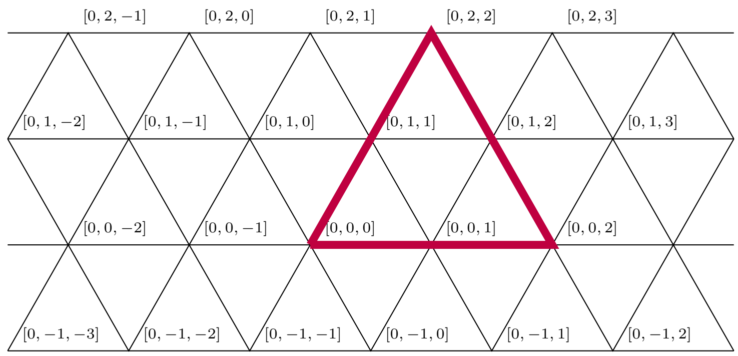

and $t(P_2) \equiv 2 \pmod 3$, we must have $t(\xi) \equiv 2 \equiv 1 \pmod 3$, which is a contradiction. Therefore, $\mathcal{N}(\Gamma_3)/D^\times\Gamma_3^\times \cong \{e\}$ is trivial.

*Remark* 3.7.7. When we talk about the symmetries of $C_\Gamma$, we mean more than symmetries of a convex polytope in a Euclidean space, as we want to preserve the underlying algebraic structure of the building. Indeed, consider the tiled order $\Gamma_4$ with exponent matrix $M_{\Gamma_4} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}$ and polytope in Figure 3.10. While the symmetry group of an equilateral triangle in $\mathbb{R}^2$ is $S_3$, the symmetry group of $C_{\Gamma_4}$ is strictly smaller, since it does not have any transposition permuting two vertices while fixing the third. Suppose there was some $\xi \in GL_3(D)$ interchanging $[P_1]$ and $[P_2]$, and fixing $[P_3]$. Then

$$t(\xi P_1) \equiv t(P_2) \pmod 3$$
$$t(\xi P_2) \equiv t(P_1) \pmod 3$$
$$t(\xi P_3) \equiv t(P_3) \pmod 3.$$

Then $t(\xi) \equiv 0 \pmod 3$, so $t(P_1) \equiv t(P_2) \pmod 3$, which contradicts that the types of $P_1$ and $P_2$ are in fact different!

Figure 3.10:




Recall that the structural invariants encode the isomorphism class of a tiled order, along with the congruence class of its associated polytope. We can find monomial representatives for the symmetries of $C_\Gamma$ for a given tiled order $\Gamma$ from its structural invariants as follows:

**Proposition 3.7.8.** *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a tiled order, and $\{m_{ij\ell} \,|\, i,j,\ell \le n\}$ its set of structural invariants. Then $\mathcal{N}(\Gamma) = \bigcup_{\sigma\in H} \xi_\sigma D^\times \Gamma^\times$, where $H$ is the subgroup of the symmetric group $S_n$ given by $H = \{\sigma \in S_n \,|\, m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i,j,\ell \le n\}$, and $\xi_\sigma = (\boldsymbol{\pi}^{\mu_{i1}-\mu_{\sigma(i)\sigma(1)}}\delta_{\sigma(i)j})$ for $\delta_{ij}$ the Kronecker delta.*

*In addition, if $\Gamma$ has full geometric rank, let $\phi : \mathcal{N}(\Gamma) \to S_n$ be the homomorphism defined in Proposition 3.7.4. Then $\mathcal{N}(\Gamma) = \bigsqcup_{\sigma\in H} \xi_\sigma D^\times \Gamma^\times$, and $\phi(\xi_\sigma) = \sigma$.*

*Proof.* First, note that $H$ is clearly a subgroup of $S_n$. Suppose we have $\sigma \in H$. Setting $\xi_\sigma := (\boldsymbol{\pi}^{\alpha_i}\delta_{\sigma(i)j})$ for $\alpha_i = \mu_{i1} - \mu_{\sigma(i)\sigma(1)}$, we get $\xi_\sigma \Gamma \xi_\sigma^{-1} = (\mathfrak{p}^{\mu'_{ij}})$, where $\mu'_{ij} = \alpha_i - \alpha_j + \mu_{\sigma(i)\sigma(j)}$. Then

$$\mu'_{ij} = \mu_{i1} - \mu_{\sigma(i)\sigma(1)} - \mu_{j1} + \mu_{\sigma(j)\sigma(1)} + \mu_{\sigma(i)\sigma(j)} = \mu_{ij} - m_{ij1} + m_{\sigma(i)\sigma(j)\sigma(1)} = \mu_{ij},$$

so $\Gamma = \xi_\sigma \Gamma \xi_\sigma^{-1}$ and $\xi_\sigma \in \mathcal{N}(\Gamma)$. Therefore, $\bigcup_{\sigma\in H} \xi_\sigma D^\times \Gamma^\times \subseteq \mathcal{N}(\Gamma)$.

To prove the other containment, let $\xi \in \mathcal{N}(\Gamma)$. By Theorem 3.2.6 and Corollary 3.2.10, we have a monomial matrix $\eta = (\boldsymbol{\pi}^{\beta_i}\delta_{\sigma(i)j}) \in \mathcal{N}(\Gamma)$ that permutes the distinguished vertices the same way $\xi$ does, so $\eta D^\times\Gamma^\times = \xi D^\times \Gamma^\times$. Since $\eta\Gamma\eta^{-1} = \Gamma$, we have that $\mu_{ij} = \beta_i - \beta_j + \mu_{\sigma(i)\sigma(j)}$ and

$$m_{ij\ell} = \mu_{ij}+\mu_{j\ell}-\mu_{i\ell} = \beta_i-\beta_j+\mu_{\sigma(i)\sigma(j)}+\beta_j-\beta_\ell+\mu_{\sigma(j)\sigma(\ell)}-\beta_i+\beta_\ell-\mu_{\sigma(i)\sigma(\ell)} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$$

for all $i, j, \ell \leq n$. By the previous paragraph, $\xi_\sigma = (\boldsymbol{\pi}^{\mu_{i1}-\mu_{\sigma(i)\sigma(1)}}\delta_{\sigma(i)j})$ is in $\mathcal{N}(\Gamma)$. But since $\eta e_{ii}\eta^{-1} = e_{\sigma^{-1}(i)\sigma^{-1}(i)} = \xi_\sigma e_{ii}\xi_\sigma^{-1}$, both $\xi_\sigma$ and $\eta$ must act on the distinguished vertices the same way, so $\xi D^\times\Gamma^\times = \eta D^\times\Gamma^\times = \xi_\sigma D^\times\Gamma^\times$, which shows the reverse containment.

Finally, suppose $\Gamma$ is has full geometric rank, and let $\sigma, \tau \in H$ such that $\xi_\sigma D^\times\Gamma^\times = \xi_\tau D^\times\Gamma^\times$. Then $\xi_\sigma\xi_\tau^{-1}e_{ii}\xi_\tau\xi_\sigma^{-1} = e_{\tau\sigma^{-1}(i)\tau\sigma^{-1}(i)}$, which means that $\xi_\sigma\xi_\tau^{-1}[P_i] = [P_{\tau\sigma^{-1}(i)}]$. Since all distinguished vertices are distinct, this only happens if $\sigma = \tau$. Finally, it is clear that $\phi(\xi_\sigma) = \sigma$. □

*Remark* 3.7.9. By Lemma 3.3.3, $[P_r] = [P_s]$ if and only if $m_{ijr} = m_{ijs}$, $m_{irj} = m_{isj}$ and $m_{rij} = m_{sij}$ for all $i, j \leq n$. Therefore, $(rs) \in H = \{\sigma \in S_n \,|\, m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq n\}$. Consider the subgroup $G := \langle (rs) : r \neq s, [P_r] = [P_s]\rangle \leq S_n$. Then $G$ is normal in $H$: let $(rs)$ a generator for $G$ as above and $\sigma \in H$, then $\sigma(rs)\sigma^{-1} = (\sigma(r)\sigma(s))$. Since $\sigma \in H$, so is $\sigma^{-1}$, and by Lemma 3.3.2 $m_{\sigma(s)\sigma(r)\sigma(s)} = m_{srs} = 0$. Therefore, $[P_{\sigma(r)}] = [P_{\sigma(s)}]$ and $(\sigma(r)\sigma(s)) \in G$.

Recall the homomorphisms $\phi : \mathcal{N}(\Gamma) \to S_t$ from Proposition 3.7.4, where $t = \#$ of distinct distinguished vertices, and $\psi : \mathcal{N}(\Gamma) \to S_n$ from Corollary 3.2.8. By

Proposition 3.7.8, $\mathrm{im}(\psi) = H$. Since $G$ identifies when $[P_r] = [P_s]$ for $r \neq s$, Lemma 3.2.9 implies that the quotient $H/G$ is isomorphic to the subgroup in $S_t$ given by $\phi(\mathcal{N}(\Gamma))$, and we have the following diagram:

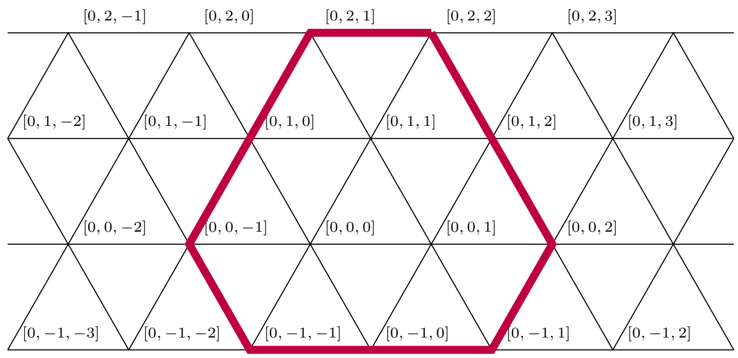
$$\begin{array}{ccc} \mathcal{N}(\Gamma) & \xrightarrow{\ \psi\ } & H \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow \\ \mathrm{im}(\phi) & \xleftrightarrow{\ \sim\ } & H/G \end{array} .$$

In particular, $\mathcal{N}(\Gamma) = \bigsqcup_\sigma \xi_\sigma D^\times \Gamma^\times$, where we take the union over a full set of coset representatives for $H/G$. However, working with the full group $H$ is more straightforward than working with the quotient $H/G$, and the full set of monomial representatives $\widetilde{H} := \{\xi_\sigma : \sigma \in H\}$ will suffice for our future purposes. We have previously referred to $\mathrm{im}(\phi)$ in $S_t$ as the symmetries of $C_\Gamma$. We will now extend the term to refer to $\sigma \in H$.

**Example 3.7.10.** Let us consider the tiled order $\Gamma$ with exponent matrix $M_\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}$. Then $m_{123} = m_{231} = m_{312} = 1$, $m_{213} = m_{321} = m_{132} = 2$, and the permutation $\sigma = (123)$ gives $m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$. A monomial representative for (123) is given by $\xi_{(123)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \boldsymbol{\pi} & 0 & 0 \end{pmatrix}$, and a representative for $\sigma^2 = (132)$ is $\xi_{(132)} = \begin{pmatrix} 0 & 0 & 1 \\ \boldsymbol{\pi} & 0 & 0 \\ 0 & \boldsymbol{\pi} & 0 \end{pmatrix}$. Note that $\xi_{(123)}\xi_{(132)} = \boldsymbol{\pi} I_3 \in D^\times\Gamma^\times$. Therefore, we have that $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong A_3$ or $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong S_3$. However, no transposition $\tau \in S_3$ gives $m_{ij\ell} = m_{\tau(i)\tau(j)\tau(\ell)}$, so $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong A_3$. We can confirm the symmetries from inspecting the polytope $C_\Gamma$ in Figure 3.11.

**Corollary 3.7.11.** *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a Gorenstein tiled order with full geometric rank, and $\sigma$ its Kirichenko permutation. Then $\sigma$ is a symmetry of $C_\Gamma$.*

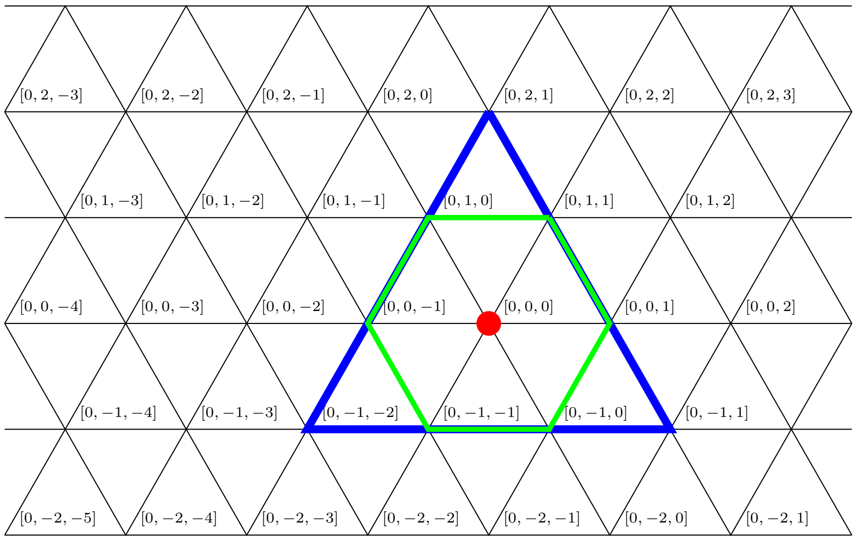
Figure 3.11:




*Proof.* This follows from [49] (also see [22, Lemma 2.1]), which states that $m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$. □

We now discuss normalizers of orders in the radical idealizer chain of $\Gamma$. Let $\xi_\sigma \in \mathcal{N}(\Gamma)$ be one of the monomial matrices defined in Proposition 3.7.8. A computation similar to the one in the proof of Proposition 3.3.6 reveals that $\xi_\sigma J(\Gamma)\xi_\sigma^{-1} = J(\Gamma)$, and $\xi_\sigma$ permutes the vertices $[Q_i] = [\widetilde{\mu}_{1i}, \widetilde{\mu}_{2i}, \ldots, \widetilde{\mu}_{ni}]$ corresponding to the columns of $M_{J(\Gamma)}$. Recall from Proposition 3.5.6 that $O_L(J(\Gamma)) = \cap_{i=1}^n \Lambda_i$, where $\Lambda_i = \mathrm{End}_R(Q_i)$. Since $\xi_\tau$ permutes the vertices $[Q_i]$, it will permute the maximal orders $\Lambda_i$, and $\xi_\sigma O_L(J(\Gamma))\xi_\sigma^{-1} = O_L(J(\Gamma))$, so $\xi_\sigma \in \mathcal{N}(O_L(J(\Gamma)))$. Reiterating the same argument, we get that any symmetry of $C_\Gamma$ will also be a symmetry of $C_{\Gamma_i}$ for each $\Gamma_i$ in the radical idealizer chain. However, the structure of $\mathcal{N}(\Gamma_i)/D^\times\Gamma_i^\times$ as a group does not get preserved. For example, in the chain depicted in Figure 3.12, we get $\mathcal{N}(\Gamma_1)/D^\times\Gamma_1^\times \cong A_3$, while $\mathcal{N}(\Gamma_2)/D^\times\Gamma_2^\times \cong S_3$ and $\mathcal{N}(\Gamma_3)/D^\times\Gamma_3^\times \cong \{e\}$.

Figure 3.12:




Section 3.8

## Normalizers of tiled orders and automorphisms of quivers

In this section, which is relatively independent from the rest of the thesis (except for Theorem 3.9.8), we assume our algebra is $M_n(k)$, where $k$ is a non-archimedean local field, with valuation ring $R$, unique maximal ideal $\mathfrak{p}$ and uniformizer $\pi$. We want to interpret the normalizer of $\Gamma$ as automorphisms of a quiver associated to $\Gamma$.

There are two equivalent constructions of such a quiver. We start with the *link graph* of $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ as defined by Müller in [27]. Let $M_1, M_2, \ldots, M_n$ be the maximal 2-sided ideals of $\Gamma$, where the $\ell^{\text{th}}$ ideal has the same exponents as $\Gamma$, except for the $(i, i)$ position, where the exponent 0 is replaced by 1 (so $M_\ell = (\mathfrak{p}^{r_{ij}})$ where $r_{ij} = \mu_{ij}$ if $\ell \neq i, j$, and $r_{\ell\ell} = 1$). The vertices of the link graph are labeled by the set $\{1, 2, \ldots, n\}$, and there is an arrow $i \to j$ when $M_j M_i \neq M_j \cap M_i$. We may also assign a value to the arrow $i \to j$ by $v(i \to j) = \mu_{ij}$. We denote the *unvalued quiver* by $Q(\Gamma)$, and the *valued quiver* by $Q_v(\Gamma)$. We denote by $[Q(\Gamma)]$ the adjacency matrix of $Q(\Gamma)$, where the $(i, j)$ entry is 0 if there is no arrow $i \to j$, and 1 if there is such an arrow.

An equivalent way to define the quiver is given by Wiedemann and Roggenkamp in [46]. Recall the ideals $\Gamma e_{ii} \subset \Gamma$, corresponding to the $i$th column of $\Gamma$ and the $i$th distinguished vertex. The radical of $\Gamma e_{ii}$ is given by $\text{rad}(\Gamma e_{ii}) = J(\Gamma) e_{ii}$, the $i$th column of the Jacobson radical of $\Gamma$. The vertices of the quiver are again labeled by $\{1, 2, \ldots, n\}$, and there is an arrow from $i$ to $j$ if and only if $\Gamma e_{ii}$ is a summand of the projective cover of $\text{rad}(\Gamma e_{ii})$.

The two constructions are equivalent by [10, Proposition 1.2]. However, note that the arrows in the two constructions are pointed in opposite directions. We will follow the convention in Müller [27]. There are a few ways to compute the adjacency matrix.

**Lemma 3.8.1** (Fujita, Oshima [12])**.** *Given a tiled order $\Gamma$, there is an arrow $i \to j$ in $Q_v(\Gamma)$ if $m_{j\ell i} > 0$ for all $\ell \neq i, j$, and there is an arrow $i \to i$ if $m_{i\ell i} > 1$ for all $\ell \neq i$.*

*Proof.* See [12, page 578]. However, note that Fujita and Oshima follow a convention where the arrows are pointed in the opposite direction. □

There is another way to find the adjacency matrix of tiled orders with full geometric rank. Recall the Jacobson radical $J(\Gamma)$ and its exponent matrix $M_{J(\Gamma)}$, and let $J(\Gamma)^2 = J(\Gamma)J(\Gamma)$ with exponent matrix $M_{J(\Gamma)^2}$.

**Theorem 3.8.2** (Theorem 14.6.2, [18])**.** *Let $\Gamma$ be a tiled order with full geometric rank. The adjacency matrix of its quiver is $[Q(\Gamma)] = (M_{J(\Gamma)^2} - M_{J(\Gamma)})^T$.*

Note that the transpose in the theorem above is due to the two different conventions.

By Skolem-Noether, the $R$-automorphisms of $\Gamma$, denoted by $\mathrm{Aut}_R(\Gamma)$ are given by conjugation by elements $\xi \in GL_n(k)$ such that $\xi\Gamma\xi^{-1} = \Gamma$, and therefore by the normalizer $\mathcal{N}(\Gamma)$. The group of automorphisms of $Q(\Gamma)$, denoted by $\mathrm{Aut}(Q(\Gamma))$ can be identified with a subgroup of $S_n$, permuting vertices with identical incoming and outgoing arrows. By [15, Lemma 1], Haefner and Pappacena describe the automorphisms of a tiled order in terms of automorphisms of its associated quiver, mirroring Proposition 3.7.4:

**Lemma 3.8.3.** *There is a group homomorphism* $\phi : \mathrm{Aut}_R(\Gamma) \to \mathrm{Aut}(Q(\Gamma))$ *whose kernel contains the inner automorphisms* $\mathrm{Inn}(\Gamma)$*.*

In [15, Theorem 5], Haefner and Pappacena prove that $\sigma \in \mathrm{Aut}(Q(\Gamma)) \subseteq S_n$ is liftable to a an element in $\mathrm{Aut}_R(\Gamma)$ if and only if the system

$$x_i - x_j = \mu_{ij} - \mu_{\sigma(i)\sigma(j)}, \quad i < j$$

has a solution $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. But then

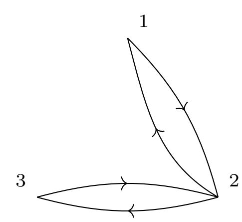
$$m_{ij\ell} - m_{\sigma(i)\sigma(j)\sigma(\ell)} = \mu_{ij} + \mu_{j\ell} - \mu_{i\ell} - \mu_{\sigma(i)\sigma(j)} - \mu_{\sigma(j)\sigma(\ell)} + \mu_{\sigma(i)\sigma(\ell)}$$

$$= (x_i - x_j) + (x_j - x_\ell) - (x_i - x_\ell) = 0,$$

which mirrors the result from Proposition 3.7.8.

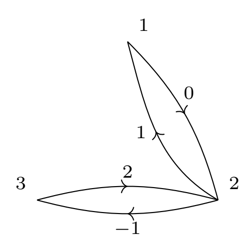
Using arguments similar to the proof of Theorem 3.2.6, they show in [15, Lemma 3] that every permutation in the image of $\phi$ has a monomial representative.

**Example 3.8.4.** Consider the tiled order $\Gamma_1$ from Example 3.7.6, with exponent matrix $M_{\Gamma_1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix}$. The unvalued quiver is given by the adjacency matrix $[Q(\Gamma_1)] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and the quiver is




The only automorphism of the quiver is permuting the first and third vertex, and from Example 3.7.6, we know there is a corresponding monomial matrix in the normalizer.

However, this automorphism of the tiled order does not correspond to an automorphism of the valued quiver $Q_v(\Gamma)$, which is given by




Usually, the group of automorphisms of the quiver is strictly larger than the image of $\phi$.

**Example 3.8.5.** Let $\Gamma$ have exponent matrix

$$M_\Gamma = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

The adjacency matrix of $Q(\Gamma)$ is given by $[Q(\Gamma)] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, so the automorphism group of the quiver is $S_3$. However, by type arguments similar to the one in Example 3.7.7, we see there is no automorphism of the tiled order interchanging the first and second vertex.

We finish the section with a few remarks about Gorenstein tiled orders. From Corollary 3.7.11 and the discussion above, the Kirichenko permutation $\sigma$ will give an automorphism of the unvalued quiver. At the same time, it follows from Theorem 3.6.4 that $m_{ij\sigma(i)} = 0$ for all $j \leq n$, which by Lemma 3.8.1 means there is no arrow from $\sigma(i) \to i$. Therefore, quivers of Gorenstein tiled orders have nice automorphisms and relatively few arrows, which make them an interesting subject of study, as seen in papers such as [35], [5] and [6].

Section 3.9

## Centered orders

In this section, we return to the general case, where the local algebra is $M_n(D)$, and $D$ is a central division algebra over a local non-archimedean field $k$. Denote by $R$ the valuation ring in $k$, by $\Delta$ the unique maximal $R$-order in $D$, and by $\boldsymbol{\pi}$ a prime element in $\Delta$ generating the unique maximal two-sided ideal $\mathfrak{p} := \boldsymbol{\pi}\Delta = \Delta\boldsymbol{\pi}$. In Proposition 3.7.8, we described representatives for the symmetries of $C_\Gamma$; in Theorem 3.9.3, we will associate to each tiled order $\Gamma \subseteq M_n(D)$ an order $\Gamma_c$ whose polytope has the same symmetries as $C_\Gamma$, but which are much easier to deduce from the exponent matrix.

In addition, we will consider $D = k$ as a special case. While the automorphisms of a tiled order $\Gamma$ in $M_n(k)$ manifest themselves as automorphisms of its quiver $Q(\Gamma)$, as we have seen in Example 3.8.4 they do not necessarily induce automorphisms of the valued quiver $Q_v(\Gamma)$. Theorem 3.9.8 will solve this inconvenience, where we will get $\mathcal{N}(\Gamma)/k^\times\Gamma^\times \cong \mathcal{N}(\Gamma_c)/k^\times\Gamma_c^\times \cong \mathrm{Aut}(Q_v(\Gamma_c))$. We start with borrowing some geometric intuition from the building theoretic context.

**Definition 3.9.1.** Let $\Gamma$ be a tiled order with exponent matrix $M_\Gamma = (\mu_{ij})$. Define $m_i := \frac{1}{n}\sum_{\ell=1}^{n} \mu_{i\ell}$ to be the average of each row. We say $\Gamma$ is **centerable** if the tuple $(0, m_2 - m_1, \ldots, m_n - m_1)$ corresponds to a vertex in the apartment. In this case, we call $[L] = [0, m_2 - m_1, \ldots, m_n - m_1]$ the **center** of $\Gamma$.

Note that the center is on $\overline{C_\Gamma}$, since

$$(m_i - m_1) - (m_j - m_1) = m_i - m_j = \frac{1}{n}\sum_{\ell=1}^{n}(\mu_{i\ell} - \mu_{j\ell}) \leq \frac{1}{n}\sum_{\ell=1}^{n}\mu_{ij} = \mu_{ij},$$

and likewise $-\mu_{ji} \leq (m_i - m_1) - (m_j - m_1)$. If we also require the centers to be fixed by all the symmetries of $C_\Gamma$, we can think of the center as a "center of symmetry", and we define a new type of tiled order:

**Definition 3.9.2.** Let $\Gamma$ be a centerable tiled order. If its center corresponds to the vertex $[L] = [0, 0, \ldots, 0]$ such that $[L]$ is fixed under all the symmetries of $C_\Gamma$, we say $\Gamma$ is **centered**.

Suppose $\sigma$ is a symmetry of $C_{\Gamma_c}$ for a centered tiled order $\Gamma_c$. Then $\xi_\sigma \in \mathcal{N}(\Gamma_c)$ must normalize the maximal order $M_n(\Delta)$ corresponding to the vertex $[0, 0, \ldots, 0]$, so $\xi_\sigma \in k^\times GL_n(\Delta)$ by Proposition 3.7.1. We can verify that $\xi_\sigma$ as defined in Proposition 3.7.8 must in fact be a permutation matrix. This gives nice relations between the exponents $\mu_{ij} = \mu_{\sigma(i)\sigma(j)}$ for all $i, j \leq n$. In the case $D = k$, $\sigma$ is an automorphism of the valued quiver $Q_v(\Gamma_c)$, since $\sigma$ is an automorphism of the unvalued quiver $Q(\Gamma_c)$ by Lemma 3.8.3, and given any arrow $i \to j$, we will have

$$\mu_{ij} = v(i \to j) = v(\sigma(i) \to \sigma(j)) = \mu_{\sigma(i)\sigma(j)}.$$

As shown later in the section, it will prove to be convenient to work with centered tiled orders. In the following theorem, given any tiled order $\Gamma$, we can associate to it a centered tiled order $\Gamma_c$ whose convex polytope $C_{\Gamma_c}$ has the same symmetries in $S_n$ as $C_\Gamma$.

**Theorem 3.9.3.** *Given a tiled order $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ with structural invariants $\{m_{ij\ell}\}$, define $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ where $\nu_{ij} = \sum_{\ell=1}^n m_{ij\ell}$. Then $\Gamma_c$ is a centered tiled order with structural invariants $\widetilde{m}_{ij\ell} = n \cdot m_{ij\ell}$ for all $1 \leq i, j, \ell \leq n$, and $\sigma \in S_n$ is a symmetry of $C_\Gamma$ if and only if $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$.*

*Proof.* First we show $\Gamma_c$ is also a tiled order. Note that

$$\nu_{ii} = \sum_{\ell=1}^{n} m_{ii\ell} = \sum_{\ell=1}^{n} (\mu_{ii} + \mu_{i\ell} - \mu_{i\ell}) = 0.$$

$\Gamma_c$ has structural invariants $\{\widetilde{m}_{ijt} \,|1 \leq i, j, t \leq n\}$ given by

$$\begin{aligned}
\widetilde{m}_{ijt} = \nu_{ij} + \nu_{jt} - \nu_{it} &= \sum_{\ell=1}^{n} m_{ij\ell} + \sum_{\ell=1}^{n} m_{jt\ell} - \sum_{\ell=1}^{n} m_{it\ell} \\
&= \sum_{\ell=1}^{n} (m_{ij\ell} + m_{jt\ell} - m_{it\ell}) \\
&= \sum_{\ell=1}^{n} (\mu_{ij} + \mu_{j\ell} - \mu_{i\ell} + \mu_{jt} + \mu_{t\ell} - \mu_{j\ell} - \mu_{it} - \mu_{t\ell} + \mu_{i\ell}) \\
&= \sum_{\ell=1}^{n} (\mu_{ij} + \mu_{jt} - \mu_{it}) = n \cdot m_{ijt} \geq 0,
\end{aligned}$$

and since $\Gamma$ itself is a tiled order and $m_{ijt} \geq 0$, it follows that $\Gamma_c$ is also a tiled order.

Next, we establish the bijection between the symmetries of $C_\Gamma$ and the elements in $S_n$ such that $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$. By Proposition 3.7.8 we need to show that

$$m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)} \quad \text{for all} \quad i,j,\ell \leq n \iff \nu_{ij} = \nu_{\sigma(i)\sigma(j)} \quad \text{for all} \quad i,j \leq n.$$

Suppose $\sigma \in S_n$ such that $m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i,j,\ell \leq n$. Then

$$\nu_{\sigma(i)\sigma(j)} = \sum_{\ell=1}^{n} m_{\sigma(i)\sigma(j)\ell} = \sum_{\sigma(\ell)=1}^{n} m_{\sigma(i)\sigma(j)\sigma(\ell)} = \sum_{\sigma(\ell)=1}^{n} m_{ij\ell} = \sum_{\ell=1}^{n} m_{ij\ell} = \nu_{ij}.$$

Conversely, if $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$, then

$$n \cdot m_{ij\ell} = \widetilde{m}_{ij\ell} = \widetilde{m}_{\sigma(i)\sigma(j)\sigma(\ell)} = n \cdot m_{\sigma(i)\sigma(j)\sigma(\ell)},$$

so by Proposition 3.7.8, $\sigma$ is a symmetry of $C_\Gamma$.

Finally, we show that $\Gamma_c$ is centered. First, we need to find its center $[L] = [0, m_2 - m_1, \ldots, m_n - m_1]$ for $m_i := \frac{1}{n}\sum_{\ell=1}^{n} \nu_{i\ell}$. Then

$$m_i = \frac{1}{n}\sum_{j=1}^{n} \nu_{ij} = \frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n}(\mu_{ij}+\mu_{j\ell}-\mu_{i\ell}) = \frac{1}{n}\left(\sum_{j=1}^{n}\sum_{\ell=1}^{n}\mu_{ij} + \sum_{j=1}^{n}\sum_{\ell=1}^{n}\mu_{j\ell} - \sum_{j=1}^{n}\sum_{\ell=1}^{n}\mu_{i\ell}\right)$$

$$= \frac{1}{n}\left(n\sum_{j=1}^{n}\mu_{ij} + \sum_{j=1}^{n}\sum_{\ell=1}^{n}\mu_{j\ell} - n\sum_{\ell=1}^{n}\mu_{i\ell}\right) = \frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n}\mu_{j\ell}.$$

Therefore $m_1 = m_2 = \cdots = m_n$, and the center is $[L] = [0, 0, \ldots, 0]$.

Now we want to show that each symmetry of $C_{\Gamma_c}$ fixes the origin. Note that since $\widetilde{m}_{ij\ell} = n \cdot m_{ij\ell}$, the symmetries of $C_\Gamma$ are the same as the symmetries of $C_{\Gamma_c}$.

Given a symmetry $\sigma \in S_n$ of $C_{\Gamma_c}$, by Proposition 3.7.8 we obtain a representative $\xi_\sigma \in \mathcal{N}(\Gamma_c)$ where $\xi_\sigma = (\pi^{\alpha_i}\delta_{\sigma(i)j})$ and $\alpha_i = \nu_{i1} - \nu_{\sigma(i)\sigma(1)} = 0$, since we have just shown that $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq n$. Therefore, $\xi_\sigma$ is a permutation matrix and $\xi_\sigma \in GL_n(\Delta)$. By Proposition 3.7.1, $\xi \in \mathcal{N}(M_n(\Delta))$. Conjugation by $\xi_\sigma$ will fix $M_n(\Delta)$, and therefore the vertex $[0, 0, \ldots, 0]$ associated to $M_n(\Delta)$. Since this holds for every symmetry of $C_{\Gamma_c}$, $\Gamma_c$ is by definition centered. $\square$

**Example 3.9.4.** Let $\Gamma$ be the tiled order with $M_\Gamma = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & -2 \\ 3 & 3 & 0 \end{pmatrix}$, with $C_\Gamma$ depicted in Figure 3.13 in blue. By Proposition 3.7.8, a representative $\xi_\sigma$ in the normalizer of $\Gamma$ is

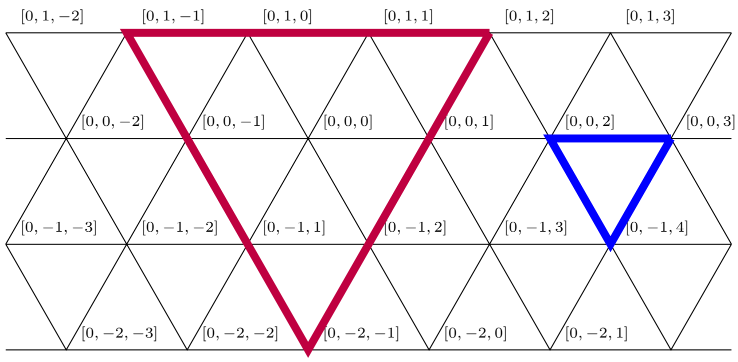
$$\xi_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \boldsymbol{\pi}^{-3} \\ \boldsymbol{\pi}^2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{\pi}^{-3} & 0 \\ 0 & 0 & \boldsymbol{\pi}^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The associated tiled order is $\Gamma_c$ with $M_{\Gamma_c} = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$, with convex polytope depicted in Figure 3.13 in red. Since $\nu_{12} = \nu_{23} = \nu_{31}, \nu_{13} = \nu_{21} = \nu_{32}$, we get a representative of the normalizer $\xi_\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

There is a geometric connection between $\Gamma$ and $\Gamma_c$, which allows us to obtain $\Gamma_c$ without necessarily finding all the structural invariants $m_{ij\ell}$:

**Corollary 3.9.5.** *Let $\Gamma$ be a tiled order. The centered tiled order $\Gamma_c$ constructed in Theorem 3.9.3 is the tiled order whose convex polytope $C_{\Gamma_c}$ is given by scaling $C_\Gamma$ by $n$, then translating the new scaled polytope so that its center aligns with the origin $[0, 0, \ldots, 0]$.*

Figure 3.13:




*Proof.* Let $\Gamma$ have exponent matrix $(\mu_{ij})$. The scaled tiled order has exponent matrix $(n\mu_{ij})$, and center $L = [\sum_{\ell=1}^{n} \mu_{1\ell}, \sum_{\ell=1}^{n} \mu_{2\ell}, \dots, \sum_{\ell=1}^{n} \mu_{n\ell}]$. The translation corresponds to conjugation by the diagonal matrix with the $(i,i)$ entry $\pi^{-\sum_{\ell=1}^{n} \mu_{i\ell}}$. The obtained tiled order has exponents

$$\nu_{ij} = n\mu_{ij} - \sum_{\ell=1}^{n} \mu_{i\ell} + \sum_{\ell=1}^{n} \mu_{j\ell} = \sum_{\ell=1}^{n} \mu_{ij} - \sum_{\ell=1}^{n} \mu_{i\ell} + \sum_{\ell=1}^{n} \mu_{j\ell} = \sum_{\ell=1}^{n} (\mu_{ij} + \mu_{j\ell} - \mu_{i\ell}) = \sum_{\ell=1}^{n} m_{ij\ell},$$

which gives us the same exponents as the statement of Theorem 3.9.3. □

*Remark* 3.9.6. Let $\Gamma$ be a tiled order with structural invariants $m_{ij\ell}$ and associated centered order $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$. By Theorem 3.9.3, $H = \{\sigma \in S_n : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq n\} = \{\sigma \in S_n : \nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq n\}$. Therefore, we can find an overgroup $G \leq S_n$ of $H$, corresponding to a product of symmetric groups, by partitioning the columns (and/or rows) of $M_{\Gamma_c}$ into sets, where each column (and/or row) has the same multiset of exponents . This is best illustrated by an example. Consider the tiled order $\Gamma$ with exponent matrix $M_\Gamma$, and using Corollary

3.9.5, we find the corresponding centered order $\Gamma_c$ with exponent matrix $M_{\Gamma_c}$.

$$M_\Gamma = \begin{pmatrix} 0 & 2 & 2 & 1 & 2 & 2 \\ 3 & 0 & 2 & 2 & 3 & 3 \\ 3 & 0 & 0 & 2 & 2 & 1 \\ 1 & 1 & 2 & 0 & 3 & 3 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 2 & 2 & 2 & 2 & 0 \end{pmatrix} \qquad M_{\Gamma_c} = \begin{pmatrix} 0 & 16 & 11 & 7 & 9 & 14 \\ 14 & 0 & 7 & 9 & 11 & 16 \\ 19 & 5 & 0 & 14 & 10 & 9 \\ 5 & 9 & 10 & 0 & 14 & 19 \\ 9 & 19 & 14 & 10 & 0 & 5 \\ 16 & 14 & 9 & 11 & 7 & 0 \end{pmatrix}.$$

Note that columns (and rows) $1, 2, 6$ have the same exponents, and so do columns (and rows) $3, 4, 5$. Therefore, setting $G = \langle (12), (26), (34), (45) \rangle \cong S_3 \times S_3$, we get $H \leq G$.

The construction of $\Gamma_c$ also helps us identify the symmetries of hereditary orders whose polytopes are chambers:

**Corollary 3.9.7.** *Let $\Gamma \subseteq M_n(D)$ be a hereditary for which $C_\Gamma$ is a chamber. Then*

$$\mathcal{N}(\Gamma)/D^\times \Gamma^\times \cong \mathbb{Z}/n\mathbb{Z}.$$

*Proof.* Since all chambers are congruent, and congruent polytopes correspond to isomorphic orders, $\Gamma$ must be isomorphic to the tiled order with exponent matrix $\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \ddots & 1 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$. Note that since all the vertices are distinct, $\mathcal{N}(\Gamma)/D^\times \Gamma^\times$ is isomorphic to a subgroup of $S_n$. By Corollary 3.9.5, $\Gamma_c$ is obtained by scaling the matrix by $n$, then translating the scaled order so that the center align with the origin. The scaling gives us the tiled order $\begin{pmatrix} 0 & n & n & \cdots & n \\ 0 & 0 & n & \cdots & n \\ 0 & 0 & 0 & \ddots & n \\ \vdots & \vdots & \vdots & \ddots & n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$, whose center is $[n-1, n-2, \ldots, 2, 1]$. The translation corresponds to conjugating the scaled order by the diagonal matrix $(\boldsymbol{\pi}^{-n+i}\delta_{ij})$. If $i < j$, the $(i, j)$ entry in the exponent matrix of $\Gamma_c$ is given by $\nu_{ij} = n+i-j$. For $i \geq j$, the exponent is $\nu_{ij} = i-j$. $\Gamma_c$ has exponent matrix $\begin{pmatrix} 0 & n-1 & n-2 & n-3 & \cdots & 1 \\ 1 & 0 & n-1 & n-2 & \cdots & 2 \\ 2 & 1 & 0 & n-1 & \cdots & 3 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ n-2 & n-3 & n-4 & \ddots & \ddots & n-1 \\ n-1 & n-2 & n-3 & n-4 & \cdots & 0 \end{pmatrix}$. Indeed, letting $\sigma = (123\ldots n)$, then $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq n$, and the only elements $\tau \in S_n$ for which $\nu_{ij} = \nu_{\tau(i)\tau(j)}$ are $\tau \in \langle\sigma\rangle$. □

We have already discussed how the symmetries of the convex polytope of a centered order give us automorphisms of its valued quiver. The following result gives us the converse as well:

**Theorem 3.9.8.** *Given a centered tiled order $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ in $M_n(k)$, there is a bijection between $\mathrm{Aut}(Q_v(\Gamma_c))$ and the symmetries of $C_{\Gamma_c}$.*

*Proof.* Let $\sigma \in \mathrm{Aut}(Q_v(\Gamma_c))$. Then $\sigma \in \mathrm{Aut}(Q(\Gamma_c))$ is also an automorphism of the unvalued quiver, and Haefner and Pappacena have shown in [15, Theorem 5] that $\sigma$ is liftable to an symmetry of $C_{\Gamma_c}$ if and only if the linear system

$$x_i - x_j = \nu_{ij} - \nu_{\sigma(i)\sigma(j)}, \quad i < j$$

has a solution $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$. Since $\sigma \in \mathrm{Aut}(Q_v(\Gamma_c))$, for any valued arrow

$\alpha : i \to j$, there is an arrow $\beta : \sigma(i) \to \sigma(j)$, and its value is $v(\beta) = v(\alpha)$. Since $v(\alpha) = \nu_{ij}$ and $v(\beta) = \nu_{\sigma(i)\sigma(j)}$, this implies $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \le n$, so the system above has a solution $(0, 0, \ldots, 0) \in \mathbb{Z}^n$. Therefore, by [15], $\sigma$ lifts to a symmetry of $C_{\Gamma_c}$.

Now suppose $\sigma$ is a symmetry of $C_{\Gamma_c}$. By [15, Lemma 1], $\sigma$ is an automorphism of the unvalued quiver $Q(\Gamma)$, so for a given arrow $\alpha : i \to j$, there is an arrow $\beta : \sigma(i) \to \sigma(j)$. To show that $\sigma$ is also an automorphism of the valued quiver $Q_v(\Gamma_c)$, we need in addition that the value of $v(\beta) = v(\alpha)$. Since $\Gamma_c$ is centered, we have from Theorem 3.9.3 that $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$. But $v(\alpha) = \nu_{ij}$ and $v(\beta) = \nu_{\sigma(i)\sigma(j)}$, so the result follows. □

Therefore, the centered tiled orders $\Gamma_c$ are quite useful, and allow us to bypass checking relations on the roughly $n^3$ structural invariants, and instead focus on $n^2$ exponents. In the final section of this chapter, we investigate possible symmetry groups for the associated polytopes $C_\Gamma$.

## Section 3.10

## Possible symmetry groups of convex polytopes

For $n = 2$, Hijikata [19] showed that if $\Gamma$ is tiled and nonmaximal (and therefore a local Eichler order), then $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong \mathbb{Z}/2\mathbb{Z}$. For $n = 3$, $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \subseteq S_3$ and in fact all subgroups of $S_3$ are realizable as symmetry groups of convex polytopes of tiled orders; we have the nontrivial subgroups in Examples 3.8.4 and 3.8.5. As $n$ increases, there are however a number of subgroups of $S_n$ that are not realizable as the symmetry group of $C_\Gamma$. In particular, we have the following easy corollary to Theorem 3.9.3:

**Corollary 3.10.1.** *Suppose we have a tiled order* $\Gamma$ *with structural invariants* $m_{ij\ell}$ *and* $H = \{\sigma \in S_n : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ *for all* $i, j, \ell \leq n\}$. *If* $H$ *is a 2-transitive subgroup of* $S_n$, *then* $H = S_n$.

*Proof.* Let $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ be the associated centered order of $\Gamma$. $H$ being 2-transitive means that given any pairs $(i, j), (\ell, t)$ with $i \neq j$ and $\ell \neq t$, there exists $\sigma \in H$ such that $\sigma(i) = \ell$ and $\sigma(j) = t$. By Theorem 3.9.3, $\nu_{ij} = \nu_{\sigma(i)\sigma(j)} = \nu_{\ell t}$. Therefore, all of the off-diagonal exponents are equal and $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $\sigma \in S_n$, so $H = S_n$. □

In particular, $A_n$ is not a possible symmetry group for $n \geq 4$, and it is the only subgroup of $S_4$ that is not realizable as a symmetry group for a tiled order $\Gamma \subseteq M_4(D)$. Once we get to $n = 5$, we get other kinds of subgroups, including subgroups that are not 2-transitive. Doing the computations in Magma [3], we got the following ineligible subgroups of $S_5$:

(a) The alternating group $A_5 = \langle (12)(35), (1, 4, 3) \rangle$ and the general affine group $GA(1, 5) = \langle (1, 2, 3, 4, 5), (2, 3, 5, 4) \rangle$ are 2-transitive.

(b) Any embedding of the alternating group $A_4$ in $S_5$ is ineligible.

(c) Any embedding of the twisted $S_3$ in $S_5$, which is a subgroup of order 6 and conjugate to $G := \langle (1, 3)(4, 5), (1, 2, 3) \rangle$. If $G \subseteq \phi(\mathcal{N}(\Gamma))$, then $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong \langle (1, 2), (1, 3), (4, 5) \rangle \cong S_3 \times S_2$.

As $n$ increases, the proportion of eligible symmetry groups decreases. When $n = 6$, 17 of the 56 subgroups of $S_6$ are ineligible. As $n \geq 10$, the proportion seems to decrease to under 40%. It would be of interest to find a way to classify the subgroups of $S_n$ that are not eligible to be symmetry groups of convex polytopes beyond the 2-transitivity criterion discussed above.

## Chapter 4

# Type numbers of global tiled orders

Section 4.1

## Introduction

In this chapter, we take a global perspective, and use the results obtained in Chapter 3 to investigate arithmetic properties of global orders. Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$. Denote the set of places of $K$ by $\mathrm{Pl}(K)$, and the subset of real places of $K$ ramifying in $A$ by $\Omega$. By the Artin-Wedderburn theorem, at each $\nu \in \mathrm{Pl}(K)$ we have $A_\nu \cong M_{r_\nu}(D_\nu)$ for some central division algebra $D_\nu$ of degree $n/r_\nu$ over $K_\nu$. Consider $\Gamma$ an order in $A$, such that $\Gamma_\nu$ is tiled at each finite place $\nu \in \mathrm{Pl}(K)$ (see the equivalent definitions in Proposition 3.1.4). If $\nu$ is an infinite place, we set $\Gamma_\nu \coloneqq A_\nu$.

By Skolem-Noether, $\Gamma$ and $\Gamma'$ are (everywhere) locally isomorphic if and only if $\Gamma_\nu = \xi_\nu \Gamma'_\nu \xi_\nu^{-1}$ for some $\xi_\nu \in A_\nu^\times$ at all finite places $\nu$ of $K$, in which case we say they are in the same *genus*. Since these local isomorphisms do not necessarily lift to a global isomorphism $\Gamma = \xi \Gamma' \xi^{-1}$ for $\xi \in A^\times$, a natural question to consider is determining the number of global isomorphism classes inside the genus of $\Gamma$. We call this number the *type number* of $\Gamma$, and will denote it by $G(\Gamma)$.

Type numbers of orders in central simple algebras have been investigated in a a few different contexts. The case for maximal and Eichler orders in definite quaternion algebras over number fields have been studied originally by Deuring [7], and subsequently by Eichler [8], Peters [30] and Pizer [31], [32]. On the other hand, type numbers of Eichler orders in not totally definite quaternion algebras have been considered by Vigneras in [44], and employs strong approximation. It is precisely this property of certain central simple algebras that we will utilize in this section to compute type numbers.

The case for algebras $A$ of degree $n \geq 3$ over a number field $K$ has been considered in [25], which investigates the genus of maximal orders. Since the degree of $A$ over $K$ is at least 3, one can apply strong approximation and express the arithmetic of the global order in terms of idelic arithmetic over the field $K$. In particular, consider a maximal order $\Lambda \subseteq A$. Recall the idelic notation for $J_K$ and $J_A$, as well as the reduced norm maps $\mathrm{nr}_{A/K} : A \to K$ and $\mathrm{nr}_{A_\nu/K_\nu} : A_\nu \to K_\nu$, which induce $\mathrm{nr} : J_A \to J_K$ where $\mathrm{nr}((a_\nu)_\nu) = (\mathrm{nr}_{A_\nu/K_\nu}(a_\nu))_\nu$. Denote the normalizer of $\Lambda_\nu$ by $\mathcal{N}(\Lambda_\nu)$, and the restricted product $\prod'_\nu \mathcal{N}(\Lambda_\nu) := J_A \cap \prod_\nu \mathcal{N}(\Lambda_\nu)$. Then the type number $G(\Lambda)$ is given by the number of double cosets $A^\times \backslash J_A / \prod'_\nu \mathcal{N}(\Lambda_\nu)$. As a consequence of strong approximation, the reduced norm induces a bijection

$$\mathrm{nr} : A^\times \backslash J_A / \prod_\nu{}' \mathcal{N}(\Lambda_\nu) \to K^\times \backslash J_K / \mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Lambda_\nu)) = J_K / K^\times \, \mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Lambda_\nu)). \quad (4.1)$$

For more details, see Chapter 34 in [34] and Section 28.4 in [45].

In this chapter, we consider non-maximal orders $\Gamma$ which are everywhere locally tiled. Since $A$ satisfies the Eichler condition, strong approximation still applies, and in order to find the number of idelic cosets, we need to find $\mathrm{nr}(\mathcal{N}(\Gamma_\nu))$ at each finite place. Proposition 4.2.2 gives an algebraic expression for $\mathrm{nr}(\mathcal{N}(\Gamma_\nu))$, which requires knowledge of the normalizer $\mathcal{N}(\Gamma_\nu)$ as described in Theorem 3. Since computing the normalizer is no trivial matter as we have seen in Chapter 3, we will use a geometric approach to find $\mathrm{nr}(\mathcal{N}(\Gamma_\nu))$. Recall that isomorphic tiled orders have congruent polytopes. Therefore, one can partition the convex polytopes congruent to $C_{\Gamma_\nu}$ into "reflection classes", where two polytopes are in the same reflection class if there exists a product of reflections sending one polytope to the other. In Theorem 4.3.7, we show that $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu)^{d_\nu}\mathcal{O}_\nu^\times$, where $d_\nu$ is determined by the number of such reflection classes of convex polytopes congruent to $C_{\Gamma_\nu}$.

There is a rather straightforward approach for finding the number of reflection classes of a polytope in a few important cases. One such case is for the local algebra $M_{r_\nu}(D_\nu)$ with $r_\nu$ prime, described in Remark 4.3.11. The general approach for finding the number of reflection classes is outlined in Remark 4.3.14.

In the final section, we compute type numbers. We express the idelic cosets in terms of class groups. An important example is given in Theorem 4.4.4, which considers algebras of odd prime degree over a field $K$; then $A_\nu \cong M_p(K_\nu)$ or $A_\nu \cong D_\nu$ a division algebra. Together with the algorithm in Remark 4.3.11, we get a powerful and straightforward approach for finding type numbers of everywhere locally tiled orders in algebras of prime degree. A general formula for type numbers is obtained in Theorem 4.4.11, which expresses type numbers as sizes of certain class group quotients.

Section 4.2

# Algebraic considerations for finding local normalizers

In this section, we investigate local reduced norms. We reestablish notation for Sections 3.2 and 3.3. Given a tiled order $\Gamma \subseteq M_r(D) =: A$, where $D$ is a central division algebra of degree $m$ over a non-archimedean local field $k$, we want to describe $\mathrm{nr}_{A/k}(\mathcal{N}(\Gamma))$. We will often lose the subscript under the reduced norm, and write $\mathrm{nr}(\mathcal{N}(\Gamma))$ instead. Denote the valuation ring of $k$ by $R$, with uniformizer $\pi$ and unique maximal ideal $P$, and the unique maximal $R$-order in $D$ by $\Delta$, with prime element $\boldsymbol{\pi} \in \Delta$ such that $\boldsymbol{\pi}^m = \pi$, and unique maximal two-sided ideal $\mathfrak{p} = \boldsymbol{\pi}\Delta$.

We start with maximal orders. For $\Lambda$ a maximal order in $M_r(D)$, by [34, (17.3)] we have $\Lambda = \xi M_r(\Delta)\xi^{-1}$ for an element $\xi \in GL_r(D)$. Recall from Corollary 3.7.2 that $\mathcal{N}(\Lambda) = \xi D^\times GL_r(\Delta)\xi^{-1}$. This allows us to deduce reduced norms:

**Lemma 4.2.1.** *Suppose $\Lambda$ is a maximal order in $A = M_r(D)$. Then*

$$\mathrm{nr}_{A/k}(\mathcal{N}(\Lambda)) = (k^\times)^r R^\times.$$

*Proof.* Since norms are multiplicative and $k$ is commutative, we have

$$\mathrm{nr}_{A/k}(\xi D^\times GL_n(\Delta)\xi^{-1}) = \mathrm{nr}_{A/k}(D^\times I_r)\,\mathrm{nr}_{A/k}(GL_r(\Delta)),$$

where $I_r$ is the $r \times r$ identity matrix. First, note that by Equation (2.2) $\mathrm{nr}_{A/k}(xI_r) = (\mathrm{nr}_{D/k}(x))^r$ for any $x \in D$. Since $\mathrm{nr}_{D/k}(D) = k$ (see page 153 in [34]), we get $\mathrm{nr}_{A/k}(D^\times I_r) = (k^\times)^r$. As discussed in Chapter 2, also $\mathrm{nr}_{D/k}(\Delta) = R$. This allows us to conclude that $\mathrm{nr}_{A/k}(GL_r(\Delta)) = R^\times$. □

Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a tiled order with structural invariants $m_{ijk}$ and normalizer $\mathcal{N}(\Gamma) = \{\xi \in \mathrm{GL}_r(D) \mid \xi\Gamma\xi^{-1} = \Gamma\}$. As seen in Chapter 3 in Proposition 3.7.8, we have $\mathcal{N}(\Gamma) = \bigcup_{\xi_\sigma \in \widetilde{H}} \xi_\sigma D^\times \Gamma^\times$, where $\widetilde{H} := \{\xi_\sigma = (\boldsymbol{\pi}^{\mu_{i1} - \mu_{\sigma(i)\sigma(1)}}) \mid \sigma \in H\}$ and $H$ is the subgroup of $S_r$ given by $H = \{\sigma \in S_r \mid m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(k)}\}$. Then

$$\mathrm{nr}_{A/k}(\mathcal{N}(\Gamma)) = \bigcup_{\xi_\sigma \in \widetilde{H}} \mathrm{nr}_{A/k}(\xi_\sigma D^\times \Gamma^\times) = \bigcup_{\xi_\sigma \in \widetilde{H}} \mathrm{nr}_{A/k}(\xi_\sigma)\, \mathrm{nr}_{A/k}(D^\times I_r)\, \mathrm{nr}_{A/k}(\Gamma^\times).$$

As before, $\mathrm{nr}_{A/k}(D^\times I_r) = (k^\times)^r$, and each maximal order $\Lambda = \zeta M_r(\Delta)\zeta^{-1}$ has $\mathrm{nr}_{A/k}(\Lambda^\times) = \mathrm{nr}_{A/k}(GL_r(\Delta)) = R^\times$. Since $\Gamma$ is an intersection of maximal orders, we get $\mathrm{nr}_{A/k}(\Gamma^\times) \subseteq R^\times$. On the other hand, the diagonal matrix $\mathrm{diag}(a, 1, 1, \ldots, 1)$ belongs to $\Gamma$ for any unit $a \in \Delta$, so $R^\times \subseteq \mathrm{nr}_{A/k}(\Gamma^\times)$. Therefore

$$\mathrm{nr}_{A/k}(\mathcal{N}(\Gamma)) = \bigcup_{\xi_\sigma \in \widetilde{H}} \mathrm{nr}_{A/k}(\xi_\sigma)(k^\times)^r R^\times.$$

Since $\xi_e = I_r$, we have $(k^\times)^r R^\times \subseteq \mathcal{N}(\Gamma)$, and we have nested subgroups

$$(k^\times)^r R^\times \subseteq \mathcal{N}(\Gamma) \subseteq k^\times.$$

The valuation $v$ on the field $k$ induces a homomorphism

$$\mathrm{nr}(\mathcal{N}(\Gamma)) \to \mathbb{Z}/r\mathbb{Z} \tag{4.2}$$

$$x \mapsto v(x) \pmod r \tag{4.3}$$

with kernel $(k^\times)^r R^\times$. Therefore $\mathrm{nr}(\mathcal{N}(\Gamma))/(k^\times)^r R^\times$ has the structure of a subgroup of $\mathbb{Z}/r\mathbb{Z}$. In particular, since $\mathrm{nr}(\mathcal{N}(\Gamma)) = \bigcup_{\xi_\sigma \in \widetilde{H}} \mathrm{nr}_{A/k}(\xi_\sigma)(k^\times)^r R^\times$, and the homomorphism above is constant on each set in this union, this subgroup is generated by the set $\{v(\mathrm{nr}(\xi_\sigma)) \pmod r : \xi_\sigma \in \widetilde{H}\}$. By Lemma 2.2.6 and the definition of the type of a matrix, we have

$$v(\mathrm{nr}(\xi_\sigma)) \equiv v_D(\det(\xi_\sigma)) \equiv t(\xi_\sigma) \pmod r.$$

Therefore, we need to find $t(\xi_\sigma)$ for $\xi_\sigma \in \widetilde{H}$. We have $\xi_\sigma[P_{\sigma(i)}] = [P_i]$ for all $i \leq r$, so $t(\xi_\sigma) \equiv t(P_i) - t(P_{\sigma(i)}) \pmod r$ for all $i \leq r$. In particular, $t(\xi_\sigma) \equiv t(P_1) - t(P_{\sigma(1)}) \pmod r$. But then $t(\xi_\tau) \equiv t(P_{\sigma(1)}) - t(P_{\tau\sigma(1)}) \pmod r$ for any other $\xi_\tau \in \widetilde{H}$. Then

$$t(\xi_{\tau\sigma}) \equiv t(P_1) - t(P_{\tau\sigma(1)}) \equiv t(\xi_\tau) + t(\xi_\sigma) \pmod r \quad \text{for all} \quad \xi_\sigma, \xi_\tau, \xi_{\sigma\tau} \in \widetilde{H}. \tag{4.4}$$

Since $H$ is a subgroup of $S_r$, the subgroup $\langle t(\xi_\sigma) : \xi_\sigma \in \widetilde{H} \rangle \subseteq \mathbb{Z}/r\mathbb{Z}$ coincides with the set $\{t(\xi_\sigma) : \xi_\sigma \in \widetilde{H}\}$. This allows us to conclude:

**Proposition 4.2.2.** *Let $\Gamma = (\mathfrak{p}^{\mu_{ij}}) \subset M_r(D)$ be a tiled order with structural invariants $m_{ij\ell}$, and $H$ the subgroup of $S_r$ where*
*$H = \{\sigma \in S_r \,|\, m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq r\}$. Let $\widetilde{H} = \{\xi_\sigma = (\boldsymbol{\pi}^{\mu_{1i} - \mu_{\sigma(1)\sigma(i)}}) : \sigma \in H\}$ be the set of monomial matrices from Proposition 3.7.8. Then*

$$\mathrm{nr}(\mathcal{N}(\Gamma)) = (k^\times)^d R^\times, \text{ where } d \,|\, r,$$

*and $d\mathbb{Z}/r\mathbb{Z} = \{t(\xi_\sigma) : \xi_\sigma \in \widetilde{H}\}$ is the subgroup of $\mathbb{Z}/r\mathbb{Z}$ given by the types of the monomial matrices in $\widetilde{H}$.*

Section 4.3

## Reflection classes

In the previous section, we have seen that $\mathrm{nr}(\mathcal{N}(\Gamma)) = (k^\times)^d R^\times$ for some $d|r$, and we have $(k^\times)^r R^\times \subseteq \mathrm{nr}(\mathcal{N}(\Gamma)) \subseteq k^\times$. In this section, we describe the factor $d$ in a building-theoretic way. Fix an apartment $\mathcal{A}$ and consider a tiled order $\Gamma$ with convex polytope $C_\Gamma$ in $\mathcal{A}$. In this section, we will define an equivalence relation between polytopes geometrically congruent to $C_\Gamma$, and will prove in Theorem 4.3.7 that $d$ is the number of equivalence classes in this equivalence relation. The first description of these classes is rather algebraic, but we have a geometric interpretation in Proposition 4.3.6.

We set notation for this section. For a tiled order $\Gamma$, we denote its structural invariants by $m_{ij\ell}$, the types of its distinguished vertices by $(t_1, \ldots, t_r) := (t(P_1), \ldots, t(P_r))$, and $H$, $\widetilde{H}$, and $d$ as in Proposition 4.2.2.

Let $\Gamma$ and $\Gamma'$ be two tiled orders, with structural invariants $(m_{ij\ell})_{i,j,\ell\leq r}$ and $(m'_{ij\ell})_{i,j,\ell\leq r}$, and types of distinguished vertices $(t_i)_{i=1}^r$ and $(t'_i)_{i=1}^r$.

**Definition 4.3.1.** Define the following relation on the set of tiled orders: $\Gamma \sim \Gamma'$ if and only if there exists $\sigma \in S_n$ such that

$$m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)} \quad \text{and} \quad t'_i = t_{\sigma(i)} \text{ for all } \; i, j, \ell \leq r.$$

The relation just defined is clearly an equivalence relation. We write

$$[\Gamma] = [(m_{ij\ell}), (t_1, t_2, \ldots, t_r)] = [(m'_{ij\ell}), (t'_1, t'_2, \ldots, t'_r)] = [\Gamma'],$$

where the tuples $(m_{ij\ell})$ and $(m'_{ij\ell})$ are in lexicographical order. By Proposition 3.3.6 two equivalent tiled orders are isomorphic, so this equivalence relation partitions convex polytopes congruent to $C_\Gamma$ in the apartment $\mathcal{A}$ into distinct classes.

**Example 4.3.2.** Let $\Gamma$, $\Gamma'$ and $\Gamma''$ have exponent matrices

$$M_\Gamma = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad M_{\Gamma'} = \begin{pmatrix} 0 & -1 & -1 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad M_{\Gamma''} = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $m_{ijj} = 0$, and $m_{iji} = m_{ij\ell} + m_{ji\ell}$, we can restrict ourselves to computing the invariants $m_{ij\ell}$ for $i \neq j \neq \ell \neq i$. Then

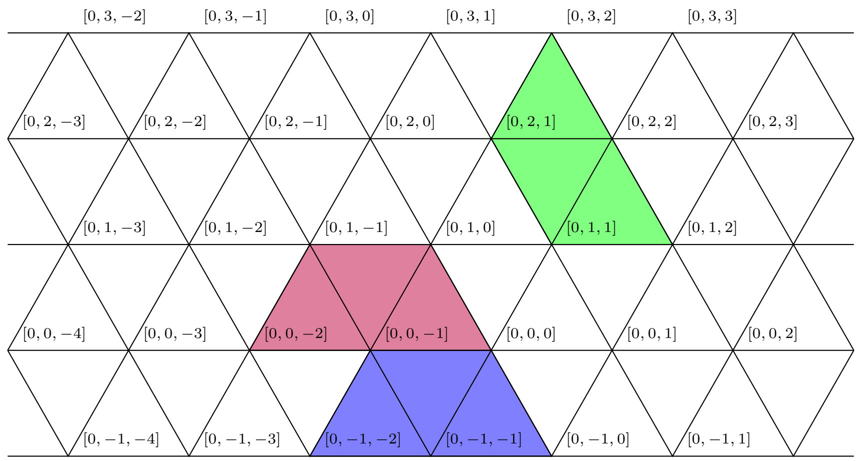
$$[\Gamma] = [(m_{123}, m_{132}, m_{213}, m_{231}, m_{312}, m_{321}), (t_1, t_2, t_3)] = [(0,2,1,1,0,1), (0,2,0)]$$
$$[\Gamma'] = [(m'_{123}, m'_{132}, m'_{213}, m'_{231}, m'_{312}, m'_{321}), (t'_1, t'_2, t'_3)] = [(1,1,1,0,0,2), (2,0,0)].$$
$$[\Gamma''] = [(m''_{123}, m''_{132}, m''_{213}, m''_{231}, m''_{312}, m''_{321}), (t''_1, t''_2, t''_3)] = [(0,2,1,1,0,1), (1,0,1)].$$

Since for $\sigma = (123)$ we have $m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ and $t'_i = t_{\sigma(i)}$, $[\Gamma] = [\Gamma'] \neq [\Gamma'']$. In Figure 4.1, $\Gamma$ has polytope in blue, $\Gamma'$ polytope in green, and $\Gamma''$ polytope in red, all three being congruent. However, $\Gamma$ and $\Gamma'$ are in the same equivalence class, while $\Gamma''$ is in a different one. We proceed by identifying a geometric criterion for determining these equivalence classes.

While studying these equivalence relations, we will make use of the following fact. Let $\Gamma = (\mathfrak{p}^{\mu_{ij}})$ be a tiled order with structural invariants $m_{ij\ell}$, distinguished vertices $[P_i]$, and types of distinguished vertices $t_i$. Given $\xi = (\boldsymbol{\pi}^{\alpha_i}\delta_{\sigma(i)j})$ a monomial matrix, and $\Gamma' = \xi\Gamma\xi^{-1} = (\mathfrak{p}^{\mu'_{ij}})$ a tiled order with structural invariants $m'_{ij\ell}$, distinguished vertices $[P'_i]$ and types $t'_i$. Then $\mu'_{ij} = \mu_{\sigma(i)\sigma(j)}$ for all $i, j \leq r$, so $\xi[P_{\sigma(i)}] = [P'_i]$ and

$$\begin{array}{rcl} m'_{ij\ell} & = & m_{\sigma(i)\sigma(j)\sigma(\ell)} \\ t'_i & = & t(\xi) + t_{\sigma(i)}. \end{array} \tag{4.5}$$

Figure 4.1:




We want to describe the equivalence relation defined above in geometric terms.

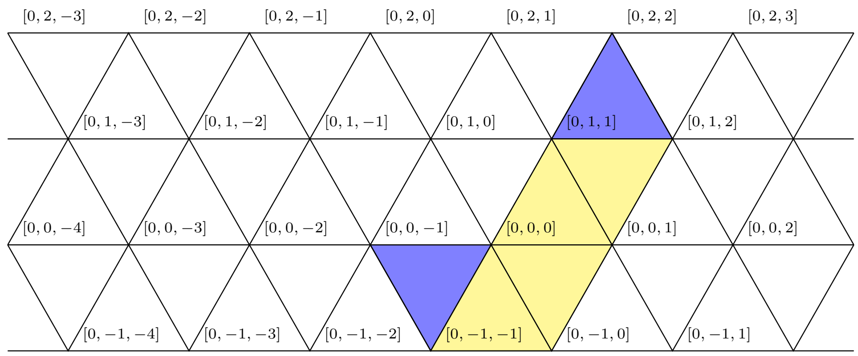
**Definition 4.3.3.** Consider two convex polytopes $C$ and $C'$ in the fixed apartment $\mathcal{A}$. We say $C$ and $C'$ are **reflection equivalent** if there is a sequence of polytopes $C = C_0, C_1, \ldots, C_s = C'$, where one can obtain each $C_{i+1}$ from $C_i$ by reflecting across some hyperplane $H$ in $\mathcal{A}$.

By definition, $C$ and $C'$ are reflection equivalent if and only if there exists a product of reflections with respect to hyperplanes in $\mathcal{A}$ sending $C$ to $C'$.

**Example 4.3.4.** Note that any two chambers are reflection equivalent. In particular, each apartment is what is called a *chamber complex* (see [14, page 52]), which means that any two chambers in the apartment are reflection equivalent (see [14, page 32]). For example, given two chambers $C$ and $C'$ outlined in blue in Figure 4.2, the sequence of chambers in yellow shows that $C$ and $C'$ are reflection equivalent.

**Example 4.3.5.** Note that $C_\Gamma$ in blue and $C_{\Gamma'}$ in green from Figure 4.1 are also

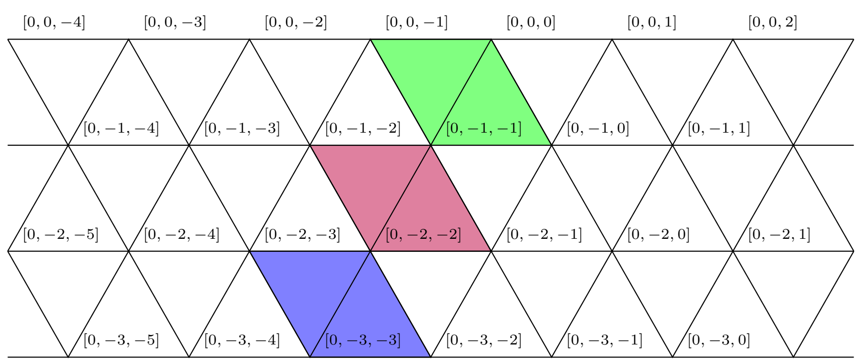
Figure 4.2:




reflection equivalent, since we can reflect $C_\Gamma$ first with respect to $x_1 - x_3 = 0$ and then with respect to $x_1 - x_2 = -1$ to obtain $C_{\Gamma'}$.

However, the three polytopes in Figure 4.3 are not reflection equivalent. Note that there is no way we can get from the polytope in blue to the one in green, or to the one in red by reflecting with respect to hyperplanes in the apartment.

Figure 4.3:




We claim the following about these equivalence classes:

**Proposition 4.3.6.** *Let $\Gamma$ and $\Gamma'$ be two isomorphic tiled orders whose convex poly-*

*topes $C_\Gamma$ and $C_{\Gamma'}$ are in $\mathcal{A}$. Then $[\Gamma] = [\Gamma']$ are in the same equivalence class from Definition 4.3.1 if and only if $C_\Gamma$ and $C_{\Gamma'}$ are reflection equivalent.*

*Proof of Proposition 4.3.6.* Suppose $[\Gamma] = [\Gamma']$, then by definition there exists $\sigma \in S_r$ such that

$$m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)} \quad \text{and} \quad t'_i = t_{\sigma(i)} \text{ for all } \; i, j, \ell \leq r.$$

We first show the claim is true when $\sigma = e$ is the identity permutation, so $t_i = t'_i$ and $m_{ij\ell} = m'_{ij\ell}$ for all $1 \leq i, j, \ell \leq r$. By Corollary 3.3.8, $C_{\Gamma'}$ must be a translation of $C_\Gamma$ by some diagonal matrix. Since $t_i = t'_i$, the type of such a diagonal matrix must be zero, and by Lemma 2.3.5 the matrix will act on the apartment by a product of reflections. This implies that $C_\Gamma$ and $C_{\Gamma'}$ are reflection equivalent.

Now suppose $\sigma \in S_r$ is nor necessarily the identity permutation. Let $M_{\Gamma'} = (\mu'_{ij})$ be the exponent matrix of $\Gamma'$, and let $\eta = (\delta_{\sigma^{-1}(i)j})$. Then $\eta$ is a permutation matrix, and therefore a product of reflections with respect to hyperplanes going through the vertex $[0, 0, \ldots, 0]$. Consider $\Gamma'' = \eta\Gamma'\eta^{-1}$. By Equation (4.5), its structural invariants and types are given by

$$m''_{ij\ell} = m'_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(\ell)} = m_{ij\ell} \qquad \text{and} \qquad t''_i = t'_{\sigma^{-1}(i)} = t_i.$$

By the previous paragraph, $C_\Gamma$ and $C_{\Gamma''}$ are reflection equivalent, and since we obtained $C_{\Gamma''}$ by reflecting $C_{\Gamma'}$ along hyperplanes going through the origin, so are $C_\Gamma$ and $C_{\Gamma'}$.

Now we prove the converse, and assume that $C_\Gamma$ and $C_{\Gamma'}$ are reflection equivalent. Let the product of reflections sending $C_\Gamma$ to $C_{\Gamma'}$ correspond to the monomial matrix

$\xi = (\boldsymbol{\pi}^{\beta_i}\delta_{\sigma(i)j})$ such that $\Gamma' = \xi\Gamma\xi^{-1}$. By Equation (4.5),

$$m'_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)} \qquad \text{and} \qquad t'_i = t_{\sigma(i)} + t(\xi) = t_{\sigma(i)},$$

and we are done. □

Therefore, the equivalence classes described above partition the convex polytopes congruent to $C_\Gamma$ into classes of reflection equivalent convex polytopes. How many such classes are there?

**Theorem 4.3.7.** *Let $\Gamma$ be a tiled order with tuple $(m_{ij\ell})$ of structural invariants in lexicographical order, and ordered tuple of types of distinguished vertices $(t_1, t_2, \ldots, t_r)$. Let $\xi_s := diag(\boldsymbol{\pi}^s, 1, \ldots, 1)$ and $\Gamma_s := \xi_s\Gamma\xi_s^{-1}$. Then there are at most $r$ reflection classes of polytopes congruent to $C_\Gamma$, corresponding to the classes of orders*

$$\begin{aligned}
[\Gamma] = [\Gamma_0] &= [(m_{ij\ell}), (t_1, t_2, \ldots, t_r)] \\
[\Gamma_1] &= [(m_{ij\ell}), (t_1 + 1, t_2 + 1, \ldots, t_r + 1)] \\
[\Gamma_2] &= [(m_{ij\ell}), (t_1 + 2, t_2 + 2, \ldots, t_r + 2)] \\
&\vdots \\
[\Gamma_{r-1}] &= [(m_{ij\ell}), (t_1 + r - 1, t_2 + r - 1, \ldots, t_r + r - 1)].
\end{aligned}$$

*In particular,* $\mathrm{nr}(\mathcal{N}(\Gamma)) = (k^\times)^d R^\times$ *if and only if there are $d$ distinct reflection classes and $[\Gamma_s] = [\Gamma_t]$ for $s \equiv t \pmod{d}$.*

*Proof.* By Equation (4.5), each $\Gamma_s$ has structural invariants and exponents

$$m'_{ij\ell} = m_{ij\ell} \qquad \text{and} \qquad t'_i \equiv t_i + s \pmod{r}.$$

We first show that any order $\Gamma'$ isomorphic to $\Gamma$, and with convex polytope $C_{\Gamma'}$ in $\mathcal{A}$, belongs to one of the classes enumerated above. If $\Gamma' \cong \Gamma$, we have $\Gamma' = \xi\Gamma\xi^{-1}$ for some monomial matrix $\xi = (\pi^{\beta_i}\delta_{\tau(i)j}), \tau \in S_r$. Let $\eta = (\delta_{\tau^{-1}(i)j})$, which is a permutation matrix and therefore corresponds to a product of reflections with respect to hyperplanes going through the origin in the apartment. Therefore, $[\Gamma'] = [\eta\Gamma'\eta^{-1}]$. Let $\Gamma'' := \eta\Gamma'\eta^{-1} = (\eta\xi)\Gamma(\eta\xi)^{-1}$. Since the product $\eta\xi$ is a diagonal matrix with $t(\eta\xi) = t(\xi)$, by Equation (4.5) $[\Gamma'']$ is determined by the data

$$m''_{ij\ell} = m_{ij\ell} \quad \text{and} \quad t''_i = t_i + t(\xi) \quad \text{for all} \quad i, j, \ell \leq r.$$

Therefore, $[\Gamma'] = [\Gamma'']$ corresponds to the reflection class given by $[\Gamma_{t(\xi)}]$.

Let $i \in \{0, 1, \ldots, d-1\}$ and $\ell \in \mathbb{Z}$ such that $t(\xi) + \ell \cdot d = i$, and let $\xi_\sigma \in \mathcal{N}(\Gamma)$ with $t(\xi_\sigma) = d$ (we know such an element exists by Proposition 4.2.2). Then $\Gamma' = \xi\xi_\sigma^\ell\Gamma\xi_\sigma^{-\ell}\xi^{-1}$, and as in the paragraph above $[\Gamma'] = [\Gamma_{t(\xi)+\ell\cdot d}] = [\Gamma_i]$, where $i \in \{0, 1, \ldots, d-1\}$. In particular, $[\Gamma_s] = [\Gamma_{s+\ell\cdot d}]$ for all $s, \ell \in \mathbb{Z}$. Therefore, there are at least $d$ reflection classes.

Now suppose we have $[\Gamma_s] = [\Gamma_t]$ for $s \leq t$ and $s, t \in \{0, 1, \ldots, d-1\}$. By Proposition 4.3.6, $[\Gamma_s] = [\Gamma_t]$ if and only if $C_{\Gamma_s}$ and $C_{\Gamma_t}$ are reflection equivalent. Since products of reflections correspond to monomial matrices of type 0, there must exist a monomial matrix $\zeta$ of type 0 such that $\zeta\Gamma_s\zeta^{-1} = \Gamma_t$. Then $\xi_s^{-1}\zeta^{-1}\xi_t \in \mathcal{N}(\Gamma)$, so $t(\xi_s^{-1}\zeta^{-1}\xi_t) = t - s \in \{0, 1, \ldots, d-1\}$. By Proposition 4.2.2, $d\mathbb{Z}/r\mathbb{Z}$ is the subgroup of $\mathbb{Z}/r\mathbb{Z}$ generated by the types of matrices in $\mathcal{N}(\Gamma)$, so $t = s$, and there are exactly $d$ distinct reflection classes. □

Theorem 4.3.7 yields the following:

**Corollary 4.3.8.** *Consider* $\Gamma$ *a tiled order in* $M_r(D)$ *with structural invariants* $m_{ij\ell}$, $H = \{\sigma \in S_r : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ *for all* $i, j, \ell \leq r\}$ *and set of monomial representatives* $\widetilde{H} = \{\xi_\sigma : \sigma \in H\}$. *Then* $[\Gamma_0] = [\Gamma_j]$ *for some* $j \leq r$ *if and only if there exists* $\sigma \in H$ *and a corresponding monomial matrix* $\xi_\sigma \in \widetilde{H}$ *such that* $t(\xi_\sigma) \equiv j \pmod r$.

*Proof.* This is a straightforward modification of the last paragraph in the proof of Theorem 4.3.7. □

**Corollary 4.3.9.** *Suppose* $\Gamma$ *is a tiled order in* $M_p(D)$, *with* $p$ *prime. Then there are either one or* $p$ *reflection classes. In particular, there can only be one reflection class if all the distinguished vertices of* $\Gamma$ *have distinct types.*

*Proof.* The first statement follows from Theorem 4.3.7. If there is only one reflection class, then $[\Gamma_0] = [\Gamma_1]$ and by Corollary 4.3.8 there must be some monomial matrix $\xi_\sigma \in \widetilde{H}$ with $t(\xi_\sigma) = 1$. By Equation (4.5) each $t_{\sigma^j(1)} \equiv t_1 + j \pmod r$, so in particular $\operatorname{ord}(\sigma) \equiv 0 \pmod r$. Since $\xi_e \in \widetilde{H}$ is the identity matrix, and $t(\xi_e) = 0$, it follows that $\sigma \neq e$ is not the identity permutation, so $\sigma$ must be a $p$-cycle, and all the distinguished vertices have distinct types. □

In the light of Theorem 4.3.7, we revisit a few examples. Consider the tiled order $\Gamma$ from Example 4.3.2 and polytope $C_\Gamma$ in blue in Figure 4.1. We see that $C_\Gamma$ has no symmetries, so $\mathcal{N}(\Gamma) = D^\times\Gamma^\times$, and therefore $\operatorname{nr}(\mathcal{N}(\Gamma)) = (k^\times)^3 R^\times$. We can check that indeed there are 3 reflection classes isomorphic to $C_\Gamma$.

Next, consider the tiled order $\Gamma$ with polytope in blue in Figure 4.3. The only non-trivial symmetry of $C_\Gamma$ is given by a reflection, which would have type 0. Therefore, $\operatorname{nr}(\mathcal{N}(\Gamma)) = (k^\times)^3 R^\times$, which agrees with the fact that there are 3 reflection classes of polytopes congruent to $C_\Gamma$.

Finally, as discussed in Example 4.3.4, any two chambers in an apartment can be connected by reflections, and all chambers in the apartment are in the same reflection class. Let $\Gamma$ be a tiled order whose polytope is a chamber; by Corollary 3.9.7, we know $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong \mathbb{Z}/r\mathbb{Z}$. Since all the vertices in a chamber have distinct types, and $\mathcal{N}(\Gamma)/D^\times\Gamma^\times$ is transitive on the distinguished vertices, it follows that there exists $\xi_\sigma \in \mathcal{N}(\Gamma)$ with $t(\xi_\sigma) \equiv 1 \pmod{r}$. By Corollary 4.3.8, we have $\mathrm{nr}(\mathcal{N}(\Gamma)) = k^\times$, which confirms that all chambers in an apartment are in the same reflection class.

The argument in the latter example and in Corollary 4.3.9 generalizes as follows.

**Corollary 4.3.10.** *Let $\Gamma \subseteq M_r(D)$ be a tiled order. Then there is only one reflection class of polytopes congruent to $C_\Gamma$ if and only if the types of the distinguished vertices of $\Gamma$ are distinct and $\mathcal{N}(\Gamma)/D^\times\Gamma^\times \cong \mathbb{Z}/r\mathbb{Z}$.*

*Proof.* Suppose there is only one reflection class of polytopes congruent to $C_\Gamma$, so in particular $[\Gamma_0] = [\Gamma_1]$. By Corollary 4.3.8, there exists $\sigma \in H$ with monomial representative $\xi_\sigma \in \widetilde{H}$ such that $t(\xi_\sigma) \equiv 1 \pmod{r}$. By Equation (4.5), $t_{\sigma(i)} = t_i + 1$ for all $i \leq r$. In particular, taking $i = 1$, we get

$$\begin{aligned} t_{\sigma(1)} &\equiv t_1 + 1 \pmod{r} \\ t_{\sigma^2(1)} &\equiv t_{\sigma(1)} + 1 \equiv t_1 + 2 \pmod{r} \\ &\vdots \\ t_{\sigma^{\mathrm{ord}(\sigma)-1}(1)} &\equiv t_1 + \mathrm{ord}(\sigma) - 1 \pmod{r} \\ t_1 &\equiv t_1 + \mathrm{ord}(\sigma) \pmod{r} \end{aligned}$$

Therefore, $r|\mathrm{ord}(\sigma)$, and the types $t_1, t_{\sigma(1)}, \ldots, t_{\sigma^{-1}(1)}$ are all distinct. Since we only have $r$ columns, $\sigma$ must be an $r$-cycle. Since the lattices corresponding to the columns of $\Gamma$ have distinct types, they must also be in distinct homothety classes, so $\mathcal{N}(\Gamma)/D^\times\Gamma^\times$ is isomorphic to a subgroup of $S_r$, and it contains the subgroup $\langle\sigma\rangle \cong \mathbb{Z}/r\mathbb{Z}$.

To finish proving the forward claim, we need to show that for all $\xi_\tau \in \mathcal{N}(\Gamma)$, we have $\tau \in \langle\sigma\rangle$. If $\xi_\tau$ fixes any vertex, it must have type zero. However, if it has type 0, and all distinguished vertices have distinct types, it must also fix all the vertices, so $\tau = e$ and therefore $\tau \in \langle\sigma\rangle$. Given any $\xi_\tau \in \mathcal{N}(\Gamma)$, let $j = \tau^{-1}(1)$. Then there exists $s \le r$ such that $\sigma^s(1) = j$. Therefore, $\tau\sigma^s(1) = 1$, and $\tau\sigma^s$ fixes the first vertex. Since $\xi_{\tau\sigma^s} \in \mathcal{N}(\Gamma)$, by the argument above $\tau\sigma^s = e$ and $\tau \in \langle\sigma\rangle$.

The converse is straightforward. Since all the distinguished vertices have distinct types, and $\mathcal{N}(\Gamma)/D^\times\Gamma^\times$ is transitive on the vertices, it follows that there exists $\xi_\sigma \in \mathcal{N}(\Gamma)$ with $t(\xi_\sigma) \equiv 1 \pmod{r}$, so $\mathrm{nr}(\mathcal{N}(\Gamma)) = k^\times$, and there is only one reflection class of polytopes congruent to $C_\Gamma$. □

We take a detour to consider the case when all distinguished vertices have distinct types, where we have a rather straightforward process for determining the number of reflection classes.

*Remark* 4.3.11. Suppose $\Gamma \subseteq M_r(D)$ is a tiled order, whose distinguished vertices have distinct types. To determine the number of reflection classes, we start with the following data:

(a) The centered order $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ as in Chapter 2. Given $\sigma \in S_r$, we know by Theorem 3.9.3 that $m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \le r$ if and only if $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \le r$.

(b) The ordered tuple of types $(t_1, t_2, \ldots, t_r)$.

(c) The proper divisors of $r$, ordered increasingly.

For each proper divisor $d$, find $\sigma \in S_r$ such that $t_i + d = t_{\sigma(i)}$. Since all $t_i$ are distinct, there exists a unique such a permutation. By definition, $[\Gamma_0] = [\Gamma_d]$ if and only if there exists $\sigma \in S_r$ such that $m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq r$, and $t_i + d = t_{\sigma(i)}$ for all $i \leq r$. Since the latter condition is satisfied, we only need to check whether $\nu_{\sigma(i)\sigma(j)} = \nu_{ij}$ for all $i, j \leq r$. If yes, return $d$. If not, choose the next divisor, and repeat the cycle.

If the algorithm does not return any divisor, there are $r$ reflection classes.

**Example 4.3.12.** Consider the tiled order $\Gamma \subseteq M_4(D)$ with exponent matrix $M_\Gamma$, and corresponding centered order $\Gamma_c$ with exponent matrix $M_{\Gamma_c}$

$$M_\Gamma = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 3 & 0 & 2 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 1 & 3 & 0 \end{pmatrix} \qquad M_{\Gamma_c} = \begin{pmatrix} 0 & 3 & 10 & 5 \\ 9 & 0 & 7 & 2 \\ 10 & 5 & 0 & 3 \\ 7 & 2 & 9 & 0 \end{pmatrix}.$$

The types are given by $(t_1, t_2, t_3, t_4) = (1, 2, 3, 4)$. We start by checking whether $[\Gamma_0] = [\Gamma_1]$. The permutation $\sigma \in S_4$ giving $t_i + 1 = t_{\sigma(i)}$ is $\sigma = (1234)$. However, $\nu_{12} \neq \nu_{23}$, so $[\Gamma_0] \neq [\Gamma_1]$. Next, we check whether $[\Gamma_0] = [\Gamma_2]$. The permutation $\tau \in S_4$ giving $t_i + 2 = t_{\tau(i)}$ is $\tau = (13)(24)$. Indeed, $\nu_{ij} = \nu_{\tau(i)\tau(j)}$ for all $i, j \leq 4$, and there must be two reflection classes congruent to $C_\Gamma$.

Note that the algorithm above avoids the issue of directly finding the normalizer, which can be quite involved as we have seen in Chapter 3. In particular, we may use the algorithm above for matrix algebras of prime degree $p$ over a division ring, since by Corollary 4.3.9 the only time we may not have $p$ distinct reflection classes is when the types are all distinct. Unfortunately, we are not quite as lucky when the types are not distinct, since then we do not have a unique $\sigma \in S_r$ permuting the types of the distinguished vertices. However, we can narrow our search to subsets of the permutation group.

**Lemma 4.3.13.** *Let $\Gamma \subseteq M_r(D)$ be a tiled order, and $\sigma \in H$ a symmetry of $C_\Gamma$ with monomial representative $\xi_\sigma$. Let $\sigma = \sigma_1\sigma_2\dots\sigma_s$ be a decomposition into disjoint cycles of length $l_1, l_2, \dots, l_s$. If any of the cycles $\sigma_i$ has length $l_i$ with $\gcd(l_i, r) = 1$, then $t(\xi_\sigma) = 0$.*

*Proof.* Note that if any of the $l_i = 1$, then $\xi_\sigma$ fixes some vertex $[P_j]$, and therefore $t(\xi_\sigma) \equiv 0 \mod r$. Therefore, we can assume $\sigma$ does not fix any $j \leq r$. Without loss of generality, suppose $\gcd(l_1, r) = 1$, and let $i \leq r$ not fixed by $\sigma_1$. Then $\sigma^{l_1} = (\sigma_2\sigma_3\dots\sigma_s)^{l_1}$ fixes $i$, so $t(\xi_{\sigma^{l_1}}) \equiv 0 \pmod r$. But $t(\xi_{\sigma^{l_1}}) \equiv l_1 t(\xi_\sigma) \pmod r$, and since $\gcd(l_1, n) = 1$, we get $t(\xi_\sigma) \equiv 0 \pmod r$. □

Therefore, the only elements in the normalizer that could contribute to make $\mathrm{nr}(\mathcal{N}(\Gamma))$ strictly larger than $(k^\times)^r R^\times$ are those that decompose into disjoint cycles of lengths which have common divisors with $r$.

A general algorithm finding the number of reflection classes runs as follows.

*Remark* 4.3.14 (Determining the number of reflection classes for $\Gamma \subset M_r(D)$). Given a tiled order $\Gamma \in M_r(D)$, we start with the following information:

(a) The centered order $\Gamma_c = (\mathfrak{p}^{\nu_{ij}})$ as in Chapter 3.

(b) The subgroup $G$ as in Remark 3.9.6. By Theorem 3.9.3, $H = \{\sigma \in S_r : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq r\}$ is the same as $\{\sigma \in S_r : \nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq r\}$, and $H \leq G$.

(c) The types $(t_1, t_2, \dots, t_r)$ of $\Gamma$.

(d) The divisors $d_i | r$ where $d_i \neq r$ in increasing order.

For each divisor $d_i$, check whether $[\Gamma_0] = [\Gamma_{d_i}]$. This is done in two steps:

(1) Recall that $[\Gamma_0] = [\Gamma_{d_i}]$ if and only if there exists $\sigma \in H$ such that $t_j + d_i = t_{\sigma(j)}$ for all $j \leq r$. Using Lemma 4.3.13, we start by finding all the permutations $\sigma \in G$ that decompose into products of disjoint cycles with length not coprime to $r$, such that $t_j + d_i = t_{\sigma(j)}$ for all $j \leq r$. If the set of such permutation is empty, take a new divisor.

(2) For each $\sigma$ found in the previous step, check whether $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq r$. If yes, then $m_{ijk} = m_{\sigma(i)\sigma(j)\sigma(\ell)}$ for all $i, j, \ell \leq r$, so we have $d$ reflection classes, and we terminate the process. If not, take a new divisor.

If the algorithm does not return any divisor, we have $r$ distinct classes.

We illustrate the algorithm with a couple of examples.

**Example 4.3.15.** Consider the tiled order $\Gamma$ with exponent matrix $M_\Gamma$, and centered order $\Gamma_c$ with exponent matrix $M_{\Gamma_c}$

$$M_\Gamma = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad M_{\Gamma_c} = \begin{pmatrix} 0 & 6 & 4 & 6 \\ 6 & 0 & 6 & 4 \\ 8 & 6 & 0 & 2 \\ 6 & 8 & 2 & 0 \end{pmatrix}.$$

The subgroup $H = \{\sigma \in S_4 : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)} \text{ for all } i, j, \ell \leq 4\}$ is contained in $G := \{e, (12), (34), (12)(34)\}$. The types are given by the tuple $(1, 3, 3, 1)$, and the proper divisors of 4 are $d_1 = 1$ and $d_2 = 2$. Since the types $t_i$ are not all distinct, by Corollary 4.3.10 we cannot have only one reflection class, so we consider the divisor $d_2 = 2$. The only permutation in $G$ decomposing into disjoint cycles of length not coprime to 4 is $\sigma = (12)(34)$, and indeed $t_i + 2 = t_{\sigma(i)}$ for all $i \leq 4$. Now we check that $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq 4$, so we must have 2 reflection classes.

**Example 4.3.16.** Consider the tiled order $\Gamma$ with exponent matrix $M_\Gamma$, and associated centered order $\Gamma_c$ with exponent matrix $M_{\Gamma_c}$

$$M_\Gamma = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{\Gamma_c} = \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}.$$

Since all the columns in $M_{\Gamma_c}$ have the same multisets of exponents, $H = \{\sigma \in S_4 : m_{ij\ell} = m_{\sigma(i)\sigma(j)\sigma(\ell)} \text{ for all } i, j, \ell \leq 4\}$ is contained in $G = S_4$. The types are given by the tuple $(0, 0, 2, 2)$, and the proper divisors of 4 are $d_1 = 1$ and $d_2 = 2$. Since the types $t_i$ are not all distinct, by Corollary 4.3.10 we cannot have only one reflection class, so we consider the divisor $d_2 = 2$. The permutations in $G$ decomposing into disjoint cycles of length not coprime to 4 such that $t_i + 2 = t_{\sigma(i)}$ are $\{(13)(24), (14)(23), (1324), (1423)\}$. Take $\sigma = (13)(24)$, then indeed $\nu_{ij} = \nu_{\sigma(i)\sigma(j)}$ for all $i, j \leq 4$, so we must have 2 reflection classes.

Section 4.4

## Computing type numbers

Now that we know how to find the number of reflection classes described in the previous section, we go back to the global setting. Recall our notation. Let $K$ be a number field with ring of integers $\mathcal{O}_K$ and set of places $\mathrm{Pl}(K)$. Let $A$ be a central simple algebra over $K$ of degree $n \geq 3$, and denote by $\Omega \subset \mathrm{Pl}(K)$ the finite set of real places of $K$ ramifying in $A$. Consider $\Gamma$ an $\mathcal{O}_K$-order in $A$, such that $\Gamma_\nu$ is tiled at each finite place $\nu \in \mathrm{Pl}(K)$. Note that at all but finitely many primes, $\Gamma_\nu$ is maximal. We denote by $K_\nu$ and $\mathcal{O}_\nu$ the completions of $K$, and respectively $\mathcal{O}_K$, at a place $\nu \in \mathrm{Pl}(K)$, and let $A_\nu := K_\nu \otimes_K A$ and $\Gamma_\nu := \mathcal{O}_\nu \otimes_R \Gamma$. By Artin-Wedderburn, $A_\nu \cong M_{r_\nu}(D_\nu)$, where $D_\nu$ is a central division algebra of degree $n/r_\nu$ over $K_\nu$. If $\nu$ is an infinite place, we set $\mathcal{O}_\nu := K_\nu$ and $\Gamma_\nu := A_\nu$.

The type number of $\Gamma$ is the number of double cosets $A^\times \backslash J_A / \prod'_\nu \mathcal{N}(\Gamma_\nu)$. Recall the bijection from Equation (4.1)

$$A^\times \backslash J_A / \prod_\nu{}' \mathcal{N}(\Gamma_\nu) \leftrightarrow J_K / K^\times \,\mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Gamma_\nu)).$$

Also, recall the idelic subsets

$$J_{K,S,\Omega} = \prod_{\nu \in \Omega} \mathbb{R}_+^\times \prod_{\nu \in S} K_\nu^\times \prod_{\nu \notin S} \mathcal{O}_\nu^\times,$$

where $S$ is a finite set of places of $K$ such that the set of infinite places $S_\infty$ is contained in $S \cup \Omega$. When $\Omega = \emptyset$, we write $J_{K,S,\Omega} = J_{K,S}$. Recall the associated class groups

$$\mathrm{Cl}_\Omega(K) \cong J_K / K^\times J_{K, S_\infty - \Omega, \Omega} \quad \mathrm{Cl}_S(K) = J_K / K^\times J_{K,S} \quad \mathrm{Cl}_{S,\Omega}(K) = J_K / K^\times J_{K,S,\Omega}.$$

### 4.4.1. Type numbers of orders in algebras with no ramified primes

We first consider the particular case in when $A$ does not ramify at any infinite place of $K$. For example, all central simple algebras of odd degree fall in this category.

Let $A := M_n(K)$. Then $A$ splits at each place $\nu \in \mathrm{Pl}(K)$, and $A_\nu \cong M_n(K_\nu)$. Let $\Gamma$ be maximal in $A$; then $\Gamma_\nu$ is maximal at each place. By Lemma 4.2.1, at each finite place $\nu$ we have $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^n \mathcal{O}_\nu^\times$. At each infinite place $\nu$, $\Gamma_\nu := A_\nu$, and since

$A$ does not ramify at any real place, then $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = A_\nu^\times$ (Theorem (33.4) in [34]). By [34, (33.4)], we get $\mathrm{nr}(\Gamma_\nu) = K_\nu^\times$.

Then

$$J_K/K^\times \,\mathrm{nr}(\prod_\nu{}' \mathcal{N}(\Gamma_\nu)) = J_K/K^\times \prod_{\nu\in S_\infty} K_\nu^\times \prod_{\nu\notin S_\infty}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times = J_K/K^\times J_K^n J_{K,S_\infty}.$$

We can obtain the type number of maximal orders in $A$ the following way:

**Proposition 4.4.1.** *The number of conjugacy classes of a maximal order $\Lambda \subseteq M_n(K)$ is*

$$G(\Lambda) = \#\,\mathrm{Cl}(K)/\,\mathrm{Cl}(K)^n.$$

*Proof.* We have $\mathrm{Cl}(K) \cong J_K/K^\times J_{K,S_\infty}$. The idelic quotient isomorphic to $\mathrm{Cl}(K)^n$ corresponds to a subgroup $H$ such that $K^\times J_{K,S_\infty} \leq H \leq J_K$. We claim that $H = K^\times J_K^n J_{K,S_\infty}$. Indeed,

$$\mathrm{Cl}(K)^n \cong J_K^n/(K^\times J_{K,S_\infty} \cap J_K^n) \cong J_K^n K^\times J_{K,S_\infty}/K^\times J_{K,S_\infty},$$

the latter isomorphism by to the second isomorphism theorem for groups.

We have

$$\mathrm{Cl}(K)/\,\mathrm{Cl}(K)^n \cong (J_K/K^\times J_{K,S_\infty})/(J_K^n K^\times J_{K,S_\infty}/K^\times J_{K,S_\infty}) \cong J_K/J_K^n K^\times J_{K,S_\infty}$$

and our claim holds. □

Now suppose $A$ is any central simple algebra that does not ramify at any infinite places, and $\Gamma$ a maximal order. At each finite place $\nu$, we get $A_\nu \cong M_{r_\nu}(D_\nu)$ for some

$r_\nu | n$ and division algebra $D_\nu$ of degree $n/r_\nu$ over $K_\nu$. Then $\Gamma_\nu$ is maximal at each finite place, and by Lemma 4.2.1, $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^{r_\nu}\mathcal{O}_\nu^\times$. We have the following condition for $G(\Lambda)$:

**Lemma 4.4.2.** *Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$ with ring of integers $\mathcal{O}_K$. Suppose that $A$ does not ramify at any infinite place of $K$. Given a maximal order $\Lambda$ in $A$, we have*

$$G(\Lambda) \leq \#\,\mathrm{Cl}(K)/\,\mathrm{Cl}(K)^n.$$

*Moreover, $G(\Lambda)$ divides $\#\,\mathrm{Cl}(K)/\,\mathrm{Cl}(K)^n$.*

*Proof.* Recall that $\#\,\mathrm{Cl}(K)/\,\mathrm{Cl}(K)^n$ is the number of isomorphism classes of a maximal order in the algebra $M_n(K)$, that is, the size of the group

$$J_K / \left[ K^\times \prod_{\nu|\infty} K_\nu^\times {\prod_{\nu\nmid\infty}}' (K_\nu^\times)^n \mathcal{O}_\nu^\times \right],$$

where by $\nu \mid \infty$ we mean the infinite places, and by $\nu \nmid \infty$ the finite ones.

On the other hand, the number of isomorphism classes of $\Lambda$ is given by the size of the group

$$J_K / \left[ K^\times \prod_{\nu|\infty} K_\nu^\times {\prod_{\nu\nmid\infty}}' (K_\nu^\times)^{r_\nu} \mathcal{O}_\nu^\times \right],$$

where each $r_\nu | n$.

Since

$${\prod_{\nu\nmid\infty}}' (K_\nu^\times)^n \mathcal{O}_\nu^\times \subseteq {\prod_{\nu\nmid\infty}}' (K_\nu^\times)^{r_\nu} \mathcal{O}_\nu^\times,$$

we have a projection from the first group onto the second, and the result follows. □

We move on to non-maximal orders. Suppose $\Gamma$ is an order in $A$ that is tiled at each finite place. In particular, $A_\nu \cong M_n(K_\nu)$ and $\Gamma_\nu \cong M_n(\mathcal{O}_\nu)$ all but finitely many places $\nu \in \mathrm{Pl}(K)$, so by Lemma 4.2.1 $\mathrm{nr}(\mathcal{N}(\Gamma)) = (K_\nu^\times)^n \mathcal{O}_\nu^\times$. At the rest of the finite places we can use Theorem 4.3.7, which states that $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times$, where $A_\nu \cong M_{r_\nu}(D_\nu)$ and $d_\nu | r_\nu$ is the number of reflection classes of polytopes congruent to $C_{\Gamma_\nu}$. Therefore, the type number of everywhere tiled orders is bounded above by the type number of maximal orders in the algebra:

**Proposition 4.4.3.** *Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$ with ring of integers $\mathcal{O}_K$. Suppose that $A$ does not ramify at any infinite places of $K$. Let $G_{max}$ be the type number of maximal orders in $A$. Given an everywhere locally tiled order $\Gamma$ in $A$, we have*

$$G(\Gamma) \leq G_{max} \leq \# \mathrm{Cl}(K)/\mathrm{Cl}(K)^n,$$

*where in particular $G(\Gamma) | G_{max}$ and $G_{max} | \# \mathrm{Cl}(K)/\mathrm{Cl}(K)^n$*

*Proof.* The proof is similar to that of Lemma 4.4.2, since at each finite place, $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times \supseteq (K_\nu^\times)^{r_\nu} \mathcal{O}_\nu^\times \supseteq (K_\nu^\times)^n \mathcal{O}_\nu^\times$, where $d_\nu | r_\nu$ and $r_\nu | n$. $\square$

Therefore, we have a few upper bounds for type numbers of locally tiled orders $\Gamma$ when the algebra $A$ is of degree $n$ over $K$, and does not ramify at any infinite place of $K$. First, it is bounded above by the type number of maximal orders in $M_n(K)$, which is equal to $\# \mathrm{Cl}(K)/\mathrm{Cl}^n(K)$. A finer bound is the type number of maximal orders in $A$.

Before we tackle the general case, we start with a simpler example. We go back to $A = M_n(K)$, and suppose either $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = K_\nu^\times$ (for example, $\Gamma_\nu$ has polytope a chamber), or $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^n \mathcal{O}_\nu^\times$ (for example, the polytope associated to $\Gamma_\nu$ has no symmetries). Then the type number $G(\Gamma)$ is given by the following class group quotient:

**Proposition 4.4.4.** *Let $A = M_n(K)$, and consider $\Gamma$ an everywhere locally tiled order in $A$ such that either $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = K_\nu^\times$, or $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^n \mathcal{O}_\nu^\times$. Denote by $T$ the union of the infinite places of $K$ and the finite places of $A$ such that $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = K_\nu^\times$, then*

$$G(\Lambda) = \# \mathrm{Cl}_T(K)/\mathrm{Cl}_T(K)^n.$$

*Proof.* We want to find $J_K/K^\times \prod_{\nu \in T} K_\nu^\times \prod_{\nu \notin T} (K_\nu^\times)^n \mathcal{O}_\nu^\times = J_K/K^\times J_{K,T} J_K^n$. Similar to finding $\mathrm{Cl}(K)^n$, we have $\mathrm{Cl}_T(K)^n \cong K^\times J_K^n J_{K,T}/K^\times J_{K,T}$, and

$$\mathrm{Cl}_T(K)/\mathrm{Cl}_T(K)^n \cong (J_K/K^\times J_{K,T})/(J_K^n K^\times J_{K,T}/K^\times J_{K,T}) \cong J_K/J_K^n K^\times J_{K,T},$$

which proves our claim. □

The proposition also allows us to easily compute type numbers of everywhere locally tiled orders in algebras of prime degree.

**Theorem 4.4.5.** *Let $A$ be a central simple algebra of prime degree $p \geq 3$ over $K$, and $\Lambda$ an everywhere locally tiled order in $A$. Let $S_\infty$ be the infinite places of $K$, $S_r$ the set of finite ramified places of $A$, and $S_t$ the set of finite places of $A$ such that the types of the distinguished vertices of $C_{\Lambda_\nu}$ are distinct and $\mathcal{N}(\Lambda_\nu)/K_\nu^\times \Lambda_\nu^\times \cong \mathbb{Z}/p\mathbb{Z}$. Given the union $T = S_\infty \cup S_r \cup S_t$, we have*

$$G(\Lambda) = \# \mathrm{Cl}_T(R)/\mathrm{Cl}_T(R)^p.$$

*Proof.* If $A$ is of prime degree $p$ over $K$, $A_\nu \cong M_p(K_\nu)$ or $A_\nu \cong D_\nu$ at each finite place $\nu \in \mathrm{Pl}(K)$, and $A_\nu \cong M_p(K_\nu^\times)$ at each infinite place. Therefore

$$\mathrm{nr}(\mathcal{N}(\Lambda_\nu)) = K_\nu^\times$$

when $\nu$ is either

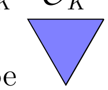
- an infinite place,

- a ramified place of $A$, or

- a place where $C_{\Lambda_\nu}$ has distinguished vertices of distinct types and $\mathcal{N}(\Lambda_\nu)/K_\nu^\times\Lambda_\nu^\times \cong \mathbb{Z}/p\mathbb{Z}$, and

$$\mathrm{nr}(\mathcal{N}(\Lambda_\nu)) = (K_\nu^\times)^p\mathcal{O}_\nu^\times \quad \text{otherwise.}$$

The rest follows from Proposition 4.4.4. □

**Example 4.4.6.** Consider $K = \mathbb{Q}(\alpha)$ where $\alpha$ is a root of $f(x) = x^3 - 7$. Then $\mathrm{Cl}(K) \cong \mathbb{Z}/3\mathbb{Z}$ is generated by the class of $\mathfrak{q} = (2, \alpha+1)$. Let $\Gamma = \begin{pmatrix} \mathcal{O}_K & \mathfrak{p} & \mathfrak{p} \\ \mathcal{O}_K & \mathcal{O}_K & \mathfrak{p} \\ \mathcal{O}_K & \mathcal{O}_K & \mathcal{O}_K \end{pmatrix}$.
Then $\Gamma_\mathfrak{p} = M_3(\mathcal{O}_\mathfrak{p})$ if $\mathfrak{p} \neq \mathfrak{q}$ and $\Gamma_\mathfrak{q} = \begin{pmatrix} \mathcal{O}_\mathfrak{q} & \mathfrak{q} & \mathfrak{q} \\ \mathcal{O}_\mathfrak{q} & \mathcal{O}_\mathfrak{q} & \mathfrak{q} \\ \mathcal{O}_\mathfrak{q} & \mathcal{O}_\mathfrak{q} & \mathcal{O}_\mathfrak{q} \end{pmatrix}$ with polytope 

The type number is the number of double cosets of

$$K^\times\backslash J_K/K_\mathfrak{q}^\times {\prod}'(K_\nu^\times)^3\mathcal{O}_\nu^\times = \mathrm{Cl}_T(K)/\mathrm{Cl}_T(K)^3,$$

for $T = S_\infty \cup \{\mathfrak{q}\}$. But $\mathrm{Cl}_T(K)$ is trivial, since any nontrivial ideal class is generated by $[\mathfrak{q}]$, and $G(\Gamma) = 1$.

To deal with the general case, we make the following observation. Weber has shown that every ideal class of $K$ is generated by some prime (see [9, p.47]). Consider an everywhere locally tiled order $\Gamma \subset A$, and let $S = \{\mathfrak{p} \in \mathrm{Pl}(K), \mathfrak{p} \notin S_\infty : \mathrm{nr}(\mathcal{N}(\Gamma_\mathfrak{p})) = (K_\mathfrak{p}^\times)^{d_\mathfrak{p}}\mathcal{O}_\mathfrak{p}^\times, d_\mathfrak{p} \neq n\}$. Since $A_\nu \cong M_n(K_\nu)$ at all but finitely many places, and $\Gamma_\nu$ is maximal at all but finitely many places, it follows that $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^n\mathcal{O}_\nu^\times$ at all but finitely many places, so $S$ is finite. For each $\mathfrak{p} \in S$, pick a prime $\mathfrak{q}_\mathfrak{p}$ generating the same ideal class as $\mathfrak{p}^{d_\mathfrak{p}}$, so $[\mathfrak{p}^{d_\mathfrak{p}}] = [\mathfrak{q}_\mathfrak{p}]$, and let $\hat{S} = \{\mathfrak{q}_\mathfrak{p} : \mathfrak{p} \in S\}$. Note that $\hat{S}$ is a finite set as well. Then

**Theorem 4.4.7.** *Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$. Suppose that $A$ does not ramify at any infinite place of $K$. Let $\Gamma$ be an order in $A$, with $S$ and $\hat{S}$ the sets of primes defined above, and denote by $T := \hat{S} \cup S_\infty$. Then*

$$G(\Gamma) = \# \mathrm{Cl}_T(K)/\mathrm{Cl}_T(K)^n.$$

*Proof.* Recall the idelic quotients from Proposition 4.4.4. We want to show that $J_K/H_1 = J_K/H_2$ where

$$H_1 := K^\times \prod_{\mathfrak{p}\in S}(K_\mathfrak{p}^\times)^{d_\mathfrak{p}}\mathcal{O}_\mathfrak{p}^\times \prod_{\nu\in S_\infty} K_\nu^\times {\prod_{\mathfrak{p}\notin S\cup S_\infty}}' (K_\mathfrak{p}^\times)^n\mathcal{O}_\mathfrak{p}^\times = \prod_{\mathfrak{p}\in S}(K_\mathfrak{p}^\times)^{d_\mathfrak{p}}(K^\times J_K^n J_{K,S_\infty})$$

$$H_2 := K^\times \prod_{\mathfrak{q}\in\hat{S}} K_\mathfrak{q}^\times \prod_{\nu\in S_\infty} K_\nu^\times {\prod_{\mathfrak{q}\notin\hat{S}\cup S_\infty}}' (K_\mathfrak{q}^\times)^n\mathcal{O}_\mathfrak{p}^\times = (\prod_{\mathfrak{q}\in\hat{S}} K_\mathfrak{q}^\times)K^\times J_K^n J_{K,S_\infty}.$$

We show that $H_1 = H_2$. For any prime $\mathfrak{p} \in S$ with associated $\mathfrak{q}_\mathfrak{p} \in \hat{S}$, and uniformizers $\pi_\mathfrak{p}$ and $\pi_{\mathfrak{q}_\mathfrak{p}}$ in $K_\mathfrak{p}$ and $K_{\mathfrak{q}_\mathfrak{p}}$, we have

$$(\ldots, 1, \pi_\mathfrak{p}^{d_\mathfrak{p}}, 1, \ldots)K^\times J_{K,S_\infty} = (\ldots, 1, \pi_{\mathfrak{q}_\mathfrak{p}}, 1, \ldots)K^\times J_{K,S_\infty},$$

so $(\ldots,1,\pi_{\mathfrak{p}}^{d_{\mathfrak{p}}},1,\ldots)K^\times J_K^n J_{K,S_\infty} = (\ldots,1,\pi_{\mathfrak{q}_{\mathfrak{p}}},1,\ldots)K^\times J_K^n J_{K,S_\infty}$. Therefore, for any integers $a_{\mathfrak{p}} \in \mathbb{Z}$ and units $u_{\mathfrak{p}} \in K_{\mathfrak{p}}$, we get

$$\prod_{\mathfrak{p}\in S}(\ldots,1,\pi_{\mathfrak{p}}^{d_{\mathfrak{p}}a_{\mathfrak{p}}}u_{\mathfrak{p}},1,\ldots)K^\times J_K^n J_{K,S_\infty} = \prod_{\mathfrak{q}_{\mathfrak{p}}\in\hat{S}}(\ldots,1,\pi_{\mathfrak{q}_{\mathfrak{p}}}^{a_{\mathfrak{p}}},1,\ldots)K^\times J_K^n J_{K,S_\infty}.$$

This implies that $H_1 \subseteq H_2$. At the same time, for any integers $b_{\mathfrak{p}} \in \mathbb{Z}$ and units $u_{\mathfrak{q}_{\mathfrak{p}}} \in \mathcal{O}_{\mathfrak{q}_{\mathfrak{p}}}$, we have

$$\prod_{\mathfrak{q}_{\mathfrak{p}}\in\hat{S}}(\ldots,1,\pi_{\mathfrak{q}_{\mathfrak{p}}}^{b_{\mathfrak{p}}}u_{\mathfrak{q}_{\mathfrak{p}}},1,\ldots)K^\times J_K^n J_{K,S_\infty} = \prod_{\mathfrak{p}\in S}(\ldots,1,\pi_{\mathfrak{p}}^{d_{\mathfrak{p}}b_{\mathfrak{p}}},1,\ldots)K^\times J_K^n J_{K,S_\infty}.$$

This implies $H_2 \subseteq H_1$, and therefore $J_K/H_1 = J_K/H_2$. □

**Example 4.4.8.** To illustrate Theorem 4.4.7 in the case $n = 4$ composite, we would like a number field $K$ for which $\mathrm{Cl}(K)/\mathrm{Cl}(K)^4$ is complicated enough, for example $\mathrm{Cl}(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$. We find such a field using the LMFDB [42]. Let $K = \mathbb{Q}(a)$ where $a$ is a root of $f(x) = x^4 - 30x^2 - 1$. Consider the order $\Gamma = \begin{pmatrix} \mathcal{O}_K & \mathfrak{p}_1 & \mathfrak{p}_1\mathfrak{p}_2 & \mathfrak{p}_1^2\mathfrak{p}_2 \\ \mathfrak{p}_1^2 & \mathcal{O}_K & \mathfrak{p}_1^2\mathfrak{p}_2 & \mathfrak{p}_1^2\mathfrak{p}_2 \\ \mathfrak{p}_1^2 & \mathfrak{p}_1 & \mathcal{O}_K & \mathfrak{p}_1 \\ \mathfrak{p}_1 & \mathfrak{p}_1 & \mathcal{O}_K & \mathcal{O}_K \end{pmatrix} \subseteq M_4(K)$, where $\mathfrak{p}_1 = (5, a+2)$ and $\mathfrak{p}_2 = (7, a-2)$. Note that $\Gamma_{\mathfrak{p}} = M_4(\mathcal{O}_{\mathfrak{p}})$ when $\mathfrak{p} \neq \mathfrak{p}_1, \mathfrak{p}_2$, and that both $\Gamma_{\mathfrak{p}_1}$ and $\Gamma_{\mathfrak{p}_2}$ are tiled. Then $\Gamma_{\mathfrak{p}_1}$ and $\Gamma_{\mathfrak{p}_2}$ have exponent matrices

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

In Examples 4.3.15 and 4.3.16, we have found that $\Gamma_{\mathfrak{p}_1}$ and $\Gamma_{\mathfrak{p}_2}$ each have two reflection classes. Therefore, we need two primes $\mathfrak{q}_1$ and $\mathfrak{q}_2$ such that $[\mathfrak{p}_1^2] = [\mathfrak{q}_1]$ and $[\mathfrak{p}_2^2] = [\mathfrak{q}_2]$.

We perform the rest of the calculations using Sage [43]. First, we find such primes $\mathfrak{q}_1 = (239, a + 36)$ and $\mathfrak{q}_2 = (7, a^3 - 33a)$. Letting $T = \{\mathfrak{q}_1, \mathfrak{q}_2\} \cup S_\infty$, we get $\mathrm{Cl}_T(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so $\mathrm{Cl}_T(K)/\mathrm{Cl}_T(K)^4 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, the type number $G(\Gamma) = 4$.

### 4.4.2. Type numbers of everywhere locally tiled orders in general central simple algebras over number fields

We generalize the results from the previous section to central simple algebras with ramification at the infinite primes. In particular, as before let $A$ be a central simple algebra of even degree $n \geq 3$ over $K$, and now let $\Omega$ be the set of real places of $K$ ramifying in $A$, so in particular $A_\nu \cong M_{n/2}(\mathbb{H})$ at each place $\nu \in \Omega$, where $\mathbb{H}$ are Hamilton's quaternions.

We summarize the idelic normalizer for the completions $\Gamma_\nu$:

$$\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = \begin{cases} (K_\nu^\times)^n R_\nu^\times & \text{for almost all } \nu \\ \mathbb{R}_+^\times & \nu \in \Omega \\ K_\nu^\times & \nu \in S_\infty - \Omega \\ (K_\nu^\times)^{d_\nu} R_\nu^\times & d_\nu \neq n, d_\nu | n. \end{cases}$$

Denote by $S := S_\infty - \Omega$, and by $T$ the set of finite places such that $\mathrm{nr}(\mathcal{N}(\Gamma_\nu)) = (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times$ where $d_\nu \neq n$. The rather complicated idelic quotient we want to find is

$$J_K / K^\times \prod_{\nu \in \Omega} \mathbb{R}_+^\times \prod_{\nu \in S} K_\nu^\times \prod_{\nu \in T} (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times \prod_{\nu \notin S_\infty \cup T}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times. \tag{4.6}$$

Proposition 4.4.3 generalizes to:

**Proposition 4.4.9.** *Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$ with ring of integers $\mathcal{O}_K$, and $\Omega$ be the set of real places ramifying in $A$. Denote by $\mathrm{Cl}_\Omega(K)$ the ray class group for $\Omega$. Let $G_{max}$ be the type number of maximal orders in $A$. Given a tiled order $\Gamma$ in $A$, we have*

$$G(\Gamma) \leq G_{max} \leq \#\,\mathrm{Cl}_\Omega(K)/\,\mathrm{Cl}_\Omega(K)^n,$$

*where in particular $G(\Gamma)|G_{max}$ and $G_{max}|\,\#\,\mathrm{Cl}_\Omega(K)/\,\mathrm{Cl}_\Omega(K)^n$.*

*Proof.* Recall that $\mathrm{Cl}_\Omega(K) \cong J_K/K^\times J_{K,S,\Omega}$, where $S = S_\infty - \Omega$. Then

$$\mathrm{Cl}_\Omega(K)^n \cong J_K^n/(K^\times J_{K,S,\Omega} \cap J_K^n) \cong J_K^n K^\times J_{K,S,\Omega}/K^\times J_{K,S,\Omega},$$

and therefore

$$\mathrm{Cl}_\Omega(K)/\,\mathrm{Cl}_\Omega(K)^n \cong (J_K/K^\times J_{K,S,\Omega})/(J_K^n K^\times J_{K,S,\Omega}/K^\times J_{K,S,\Omega}) \cong J_K/J_K^n K^\times J_{K,S,\Omega}.$$

Recall that

$$J_{K,S,\Omega} = \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu \text{ finite}} \mathcal{O}_\nu^\times,$$

and since $n$ is even,

$$J_K^n = \prod_{\nu \text{ real}} \mathbb{R}_+^\times \prod_{\nu \text{ complex}} K_\nu^\times \prod_{\nu \text{ finite}}{}' (K_\nu^\times)^n \mathcal{O}_\nu.$$

Therefore,

$$J_K^n J_{K,S,\Omega} = \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu \text{ finite}}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times.$$

Note that then

$$\mathrm{Cl}_\Omega(K)/\mathrm{Cl}_\Omega(K)^n = J_k/K^\times \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu \text{ finite}}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times$$

corresponds to the quotient in Equation (4.6) for $T = \emptyset$.

The genus of a maximal order $\Lambda$ in $A$ will correspond to the quotient

$$J_K/K^\times \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu\in T} (K_\nu^\times)^{r_\nu} \mathcal{O}_\nu^\times \prod_{\nu\notin S_\infty\cup T}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times$$

where $T$ is the set of finite primes where $A_\nu \cong M_{r_\nu}(D_\nu)$ for which $r_\nu \neq n$.

Finally, the genus of an arbitrary globally tiled order is given by

$$J_K/K^\times \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu\in T} (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times \prod_{\nu\notin S_\infty\cup T}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times.$$

Since

$$\prod_{\nu \text{ finite}}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times \subseteq \prod_{\nu\in T} (K_\nu^\times)^{r_\nu} \mathcal{O}_\nu^\times \prod_{\nu\notin S_\infty\cup T}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times \subseteq \prod_{\nu\in T} (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times \prod_{\nu\notin S_\infty\cup T}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times,$$

the result follows. □

*Remark* 4.4.10. Note that the genus of a locally tiled order $\Gamma$ is given by

$$J_K/K^\times \prod_{\nu\in\Omega} \mathbb{R}_+^\times \prod_{\nu\in S} K_\nu^\times \prod_{\nu\in T} (K_\nu^\times)^{d_\nu} \mathcal{O}_\nu^\times \prod_{\nu\notin S_\infty\cup T}{}' (K_\nu^\times)^n \mathcal{O}_\nu^\times = J_K/\prod_{\nu\in T} (K_\nu^\times)^{d_\nu} K^\times J_K^n J_{K,S,\Omega}.$$

Now we want to generalize Proposition 4.4.7. As before, let $S = S_\infty - \Omega$. By the Chebotarev density theorem, each ideal class in $\mathrm{Cl}_\Omega(K)$ is generated by some prime ideal. Consider an everywhere locally tiled order $\Gamma \subset A$, and let $T$ be the finite set of places $T = \{\mathfrak{p} \in \mathrm{Pl}(K), \mathfrak{p} \notin S_\infty : \mathrm{nr}(\mathcal{N}(\Gamma_\mathfrak{p})) = (K_\mathfrak{p}^\times)^{d_\mathfrak{p}} \mathcal{O}_\mathfrak{p}^\times, d_\mathfrak{p} \neq n\}$. For each $\mathfrak{p} \in T$, pick a prime $\mathfrak{q}_\mathfrak{p}$ generating the same ideal class as $\mathfrak{p}^{d_\mathfrak{p}}$, so $[\mathfrak{p}^{d_\mathfrak{p}}] = [\mathfrak{q}_\mathfrak{p}]$, and let $\hat{T} = \{\mathfrak{q}_\mathfrak{p} : \mathfrak{p} \in T\} \cup S$. Note that $\hat{T}$ is a finite set as well. Then

**Theorem 4.4.11.** *Let $A$ be a central simple algebra of degree $n \geq 3$ over a number field $K$, with $\Omega$ the set of real ramified primes in $A$. Let $\Gamma$ be an everywhere locally tiled order in $A$, with $T$ and $\hat{T}$ the sets of places defined above, and $S = S_\infty - \Omega$ the set of infinite paces which do not ramify in $A$. Then*

$$G(\Gamma) = \# \,\mathrm{Cl}_{\hat{T},\Omega}(K) / \,\mathrm{Cl}_{\hat{T},\Omega}(K)^n.$$

*Proof.* Recall that $\mathrm{Cl}_{\hat{T},\Omega}(K) \cong J_K / K^\times J_{K,\hat{T},\Omega}$, and similar to the formula for $\mathrm{Cl}_\Omega(K)^n$, we also have $\mathrm{Cl}_{\hat{T},\Omega}(K)^n \cong J_K^n K^\times J_{K,\hat{T},\Omega} / K^\times J_{K,S,\Omega}$, which gives

$$\mathrm{Cl}_{\hat{T},\Omega}(K) / \,\mathrm{Cl}_{\hat{T},\Omega}(K)^n \cong J_K / J_K^n K^\times J_{K,\hat{T},\Omega} = J_K / J_K^n K^\times J_{K,S,\Omega} \prod_{\mathfrak{q} := \mathfrak{q}_\mathfrak{p} : \mathfrak{p} \in T} K_\mathfrak{q}^\times.$$

Recall Remark 4.4.10. We want to show

$$J_K / K^\times J_K^n J_{K,S,\Omega} \prod_{\mathfrak{p} \in T} (K_\mathfrak{p}^\times)^{d_\mathfrak{p}} = J_K / K^\times J_K^n J_{K,S,\Omega} \prod_{\mathfrak{q} = \mathfrak{q}_\mathfrak{p} : \mathfrak{p} \in T} K_\mathfrak{q}^\times.$$

Let

$$H_1 := K^\times J_K^n J_{K,S,\Omega} \prod_{\mathfrak{p} \in T} (K_\mathfrak{p}^\times)^{d_\nu} \qquad \text{and} \qquad H_2 := K^\times J_K^n J_{K,S,\Omega} \prod_{\mathfrak{q} = \mathfrak{q}_\mathfrak{p} : \mathfrak{p} \in T} K_\mathfrak{q}^\times.$$

We show that $H_1 = H_2$. Similar to the proof of Theorem 4.4.7, for any prime

$\mathfrak{p} \in T$ with associated $\mathfrak{q}_\mathfrak{p}$, and uniformizers $\pi_\mathfrak{p}$ and $\pi_{\mathfrak{q}_\mathfrak{p}}$ in $K_\mathfrak{p}$ and $K_{\mathfrak{q}_\mathfrak{p}}$, we have $(\ldots, 1, \pi_\mathfrak{p}^{d_\mathfrak{p}}, 1, \ldots)K^\times J_{K,S,\Omega} = (\ldots, 1, \pi_{\mathfrak{q}_\mathfrak{p}}, 1, \ldots)K^\times J_{K,S,\Omega}$, which gives

$$\prod_{\mathfrak{p}\in T}(\ldots, 1, \pi_\mathfrak{p}^{d_\mathfrak{p} a_\mathfrak{p}} u_\mathfrak{p}, 1, \ldots)K^\times J_K^n J_{K,S_\infty} = \prod_{\mathfrak{q}_\mathfrak{p} : \mathfrak{p}\in T}(\ldots, 1, \pi_{\mathfrak{q}_\mathfrak{p}}^{a_\mathfrak{p}}, 1, \ldots)K^\times J_K^n J_{K,S_\infty}$$

for any integers $a_\mathfrak{p} \in \mathbb{Z}$ and units $u_\mathfrak{p} \in K_\mathfrak{p}$. This implies that $H_1 \subseteq H_2$. Similarly $H_2 \subseteq H_1$, and therefore $J_K/H_1 = J_K/H_2$. $\square$

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