McMaster University

FACULTY OF SCIENCE
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Exponential functions and Compound Interest

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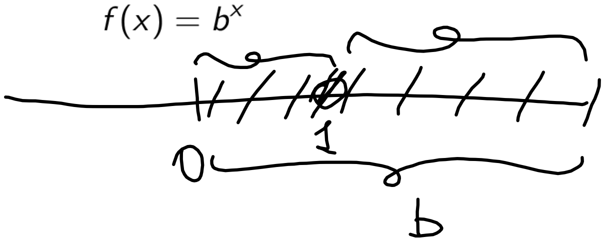
Lecture 2

Exponential functions: definition

Definition. If $b > 0$, and $b \neq 1$, the exponential function with base b is

$$f(x) = b^x$$

for every real x.



2 cases: $0<b<1$ or $b>1$

e.g: Case 1: $b>1$, e.g. $f(x)=2^x$

Table of values:

x	-8	-5	0	1	3	5	10
2^x	0.004	0.031	1	2	8	32	1024

e.g. $f(-8) = 2^{-8} = \dfrac{1}{2^8} \approx 0.004$

$f(0) = 2^0 = 1$

$f(3) = 2^3 = 2 \cdot 2 \cdot 2 = 8$

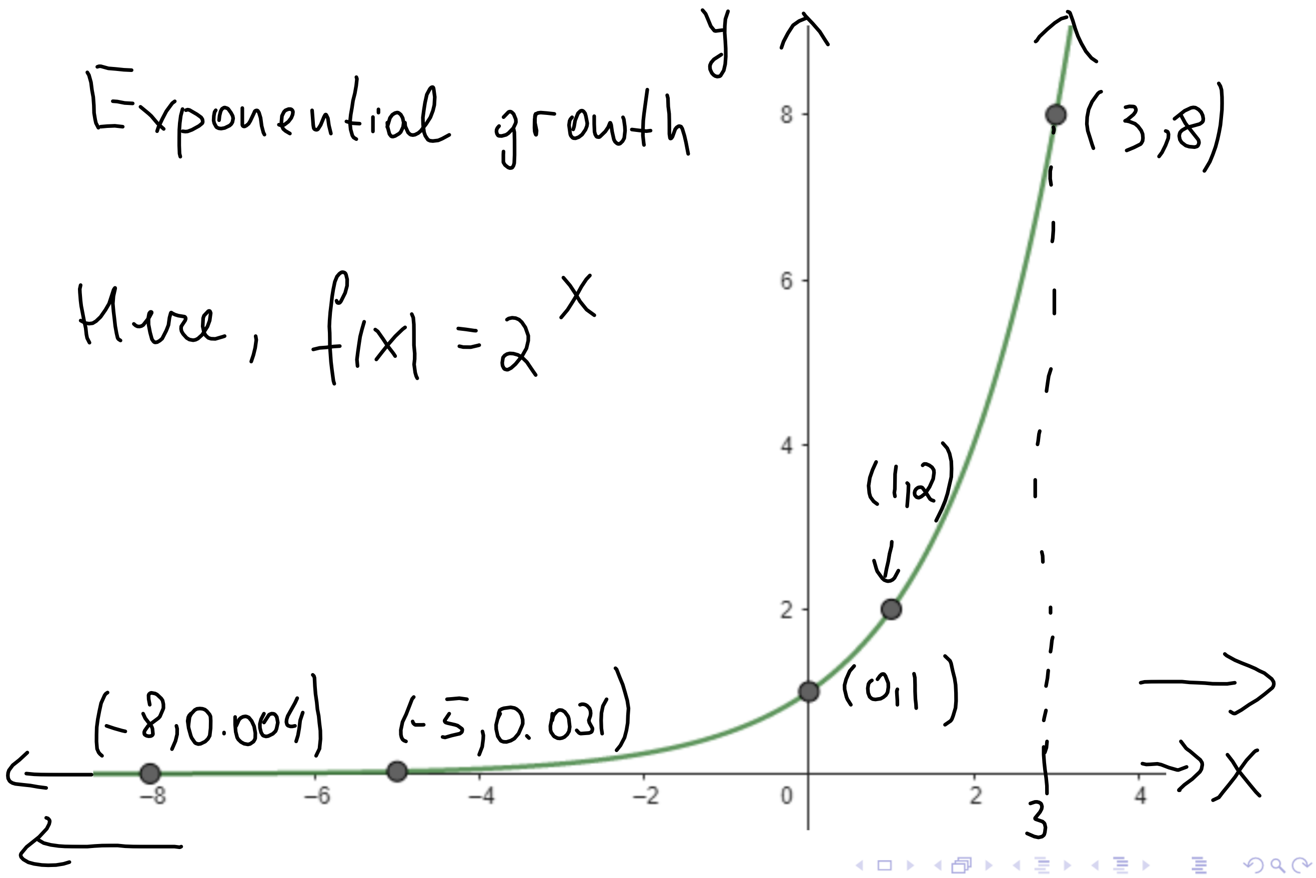
E.g: Can find 2^x for any rational $x = \dfrac{m}{n}$,

$2^{m/n} = \sqrt[n]{2^m}$ (e.g. $2^{3/4} = ?$ on calculator)

Case 1: Graph of $f(x) = b^x$ when $b > 1$

Exponential growth

Here, $f(x) = 2^x$



when $b>1$, long-term behavior

$$\lim_{x \to -\infty} b^x = 0 \qquad \lim_{x \to +\infty} b^x = +\infty$$

Case 2: $f(x) = b^x$ when $b < 1$

e.g.: $f(x) = \left(\frac{1}{2}\right)^x$

x	-10	-3	-1	0	5	8
$\left(\frac{1}{2}\right)^x$	1024	8	2	1	0.031	0.004

$$f(-10) = \left(\frac{1}{2}\right)^{-10} = \frac{1}{\left(\frac{1}{2}\right)^{10}} = \frac{1}{\frac{1}{2^{10}}}$$

flip

$$= 2^{10} = 1024$$

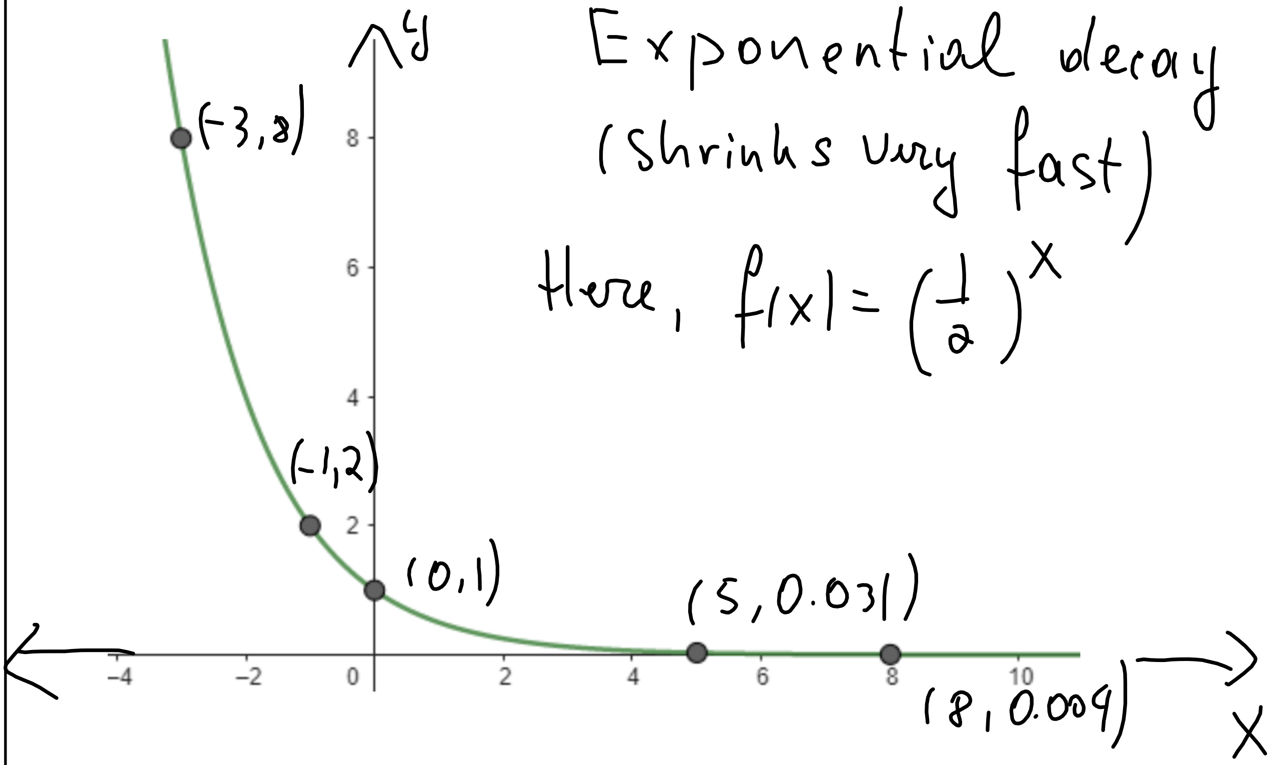
$$f(0) = \left(\frac{1}{2}\right)^0 = 1$$

$$\left(\frac{1}{2}\right)^5 = \frac{1}{2^5} \approx 0.031$$

Case 2: Graph of $f(x) = b^x$ when $\cancel{b < 1}$ $0 < b < 1$

Exponential decay
(shrinks very fast)

Here, $f(x) = \left(\frac{1}{2}\right)^x$



When $x \to -\infty$:

$$\lim_{x \to -\infty} f(x) = +\infty$$

$$\lim_{x \to +\infty} f(x) = 0$$

Take a minute to digest!

Take a minute to write as much as you can remember from the past 10 minutes.

Solving exponential ~~functions~~ equations

Example: Solve for x the equation

$$2^{30-x} = 4^x.$$

Tip: Bring both sides to the same base, then use equality rule

$$b^y = b^z, \text{ then } y = z$$

Here, $4 = 2 \cdot 2 = 2^2$, bring both sides to base 2.

$$2^{30-x} = (2^2)^x$$

power

$$2^{30-x} = 2^{2x}$$

equality

$$30 - x = 2x$$

$$30 = 2x + x$$

$$30 = 3x$$

$$\boxed{x = 10}$$

Some special bases

$$b = 2 \to \text{computer science}$$

$$b = 10 \to \text{biology, chemistry}$$

$$b = e \approx \underbrace{2.71828\cdots}_{\text{approx}} \to \text{finance}$$

Here,

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \quad \Big\}\ \text{exact}$$

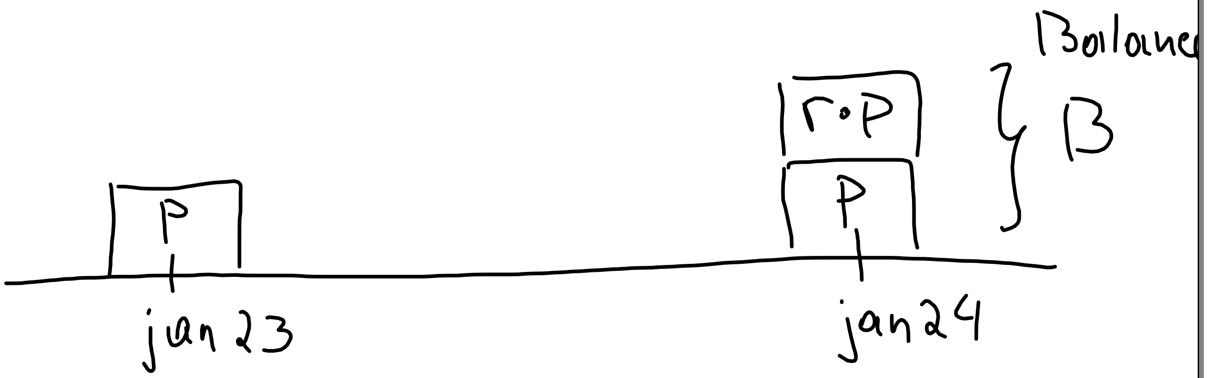
Then $f(x) = e^x$ is the "natural exponential function".
In your spare time, try finding the constant e on your calculator.

Application to finance: compound interest

Question. Given an initial investment P (called principal), and an annual interest rate r, how much money do you have at the end of the year?

Answer. Depends on how often you compound (i.e. calculate) the interest during the year.

- If compounded once (at the end of the year), we call this simple interest:



Balance after 1 year:

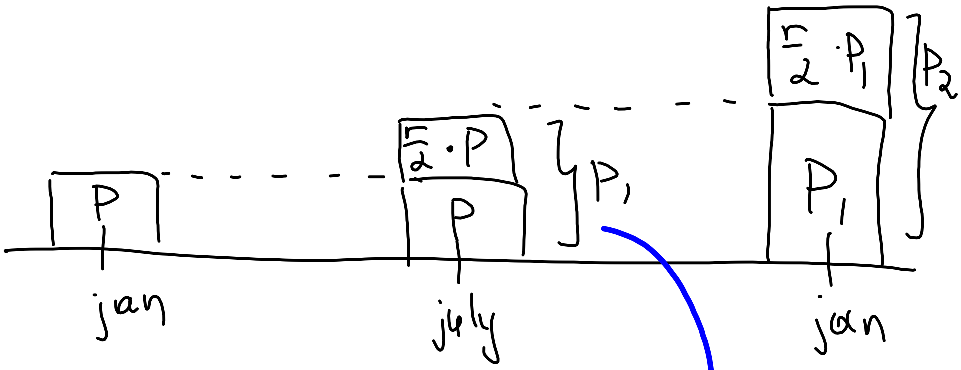
$$B = P + r \cdot P = P(1+r)$$

E.g 1: $P = \$100$, at $r = 5\%$, compounded once. $5\% = 0.05 = \frac{5}{100}$

$$B = 100(1+0.05)$$
$$= 105\ (\$)$$

Application to finance: compound interest (cont.)

- If compounded twice a year, after half a year you get half the interest, and at the end of the year you get the other half.



Balance after 1 year is

$$B = P_2 = P_1 + \frac{r}{2}P_1 = P_1\left(1+\frac{r}{2}\right)$$

$$= \left(P + \frac{r}{2}P\right)\left(1+\frac{r}{2}\right)$$

$$= P\left(1+\frac{r}{2}\right)\left(1+\frac{r}{2}\right) = P\left(1+\frac{r}{2}\right)^2$$

Ex2: $P = 100$, $r = 5\%$, compound twice.

$$B = 100\cdot\left(1+\frac{0.05}{2}\right)^2$$

$$= 105.0625 > 105 \text{ (ex 1)}$$

Idea: Calculating the interest more often works in our favor.

Application to finance: compound interest (cont.)

- If interest is calculated k times (a year), after the kth part of the year you have

$$\underbrace{P_1}_{} = P\left(1+\frac{r}{k}\right)^{\underbrace{1}_{}}$$

after another kth part

$$\underbrace{P_2}_{} = P_1\left(1+\frac{r}{k}\right) = P\left(1+\frac{r}{k}\right)^{\underbrace{2}_{}}$$

$$\underbrace{P_3}_{} = P_2\left(1+\frac{r}{k}\right) = P\left(1+\frac{r}{k}\right)^{\underbrace{3}_{}}$$

$$\vdots$$

$$B = \underbrace{P_k}_{} = P\left(1+\frac{r}{k}\right)^{\underbrace{k}_{}}$$

(after the whole year)

Application to finance: compound interest (cont.)

Therefore, the balance at the end of the year, given a (yearly) interest rate r compounded k times is

$$\boxed{B = P\left(1 + \frac{r}{k}\right)^k}$$

Now, suppose you compound interest k times a year, for t years. In t years, the number of times you calculate the interest is $k \cdot t$, and the balance after t years is given by

$$B(t) = P\left(1 + \frac{r}{k}\right)^{kt}$$