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Compound Interest and Logarithmic functions

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Lecture 3

Application to finance: compound interest (cont.)

Last time, we saw that the balance at the end of the year, given a (yearly) interest rate r compounded k times is

$$\boxed{B = P\left(1+\frac{r}{k}\right)^k}$$

Now, suppose you compound interest k times a year, for t years. In t years, the number of times you calculate the interest is $k \cdot t$, and the balance after t years is given by

$$\boxed{B(t) = P\left(1+\frac{r}{k}\right)^{kt}}$$

After 1 year: $B(1) = P\left(1+\frac{r}{k}\right)^k$

$$B(2) = P\left(1+\frac{r}{k}\right)^k \cdot \left(1+\frac{r}{k}\right)^k$$
$$= P\left(1+\frac{r}{k}\right)^{2k}$$

$\vdots$

$$B(t) = P\left(1+\frac{r}{k}\right)^{tk}$$

Continuous compounding

Recall the philosophy from the end of Example 2 from last lecture: the more times we compound, the more money we make.
Idea. What if we let the number of times we compound go to infinity? This is continuous compounding.

k times a year:

$$B(t) = P\left(1+\frac{r}{k}\right)^{kt}$$

Take $k \to \infty$

$$\lim_{k\to\infty} P\left(1+\frac{r}{k}\right)^{kt} = ?$$

Recall: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Change of variable: $\boxed{\frac{r}{k} = \frac{1}{n}}$

$k = rn$

$$\lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^n\right]^{rt} = P\cdot e^{rt}$$

So $B(t) = Pe^{rt}$

$P=100$, $r = 5\% = 0.05$

Ex 1 (last time):

$k=1$: $B(1) = 105$

Ex 2: $k=2$, then $B(1) = 105.0625$

Ex 3: continuous

then $B(1) = 100\cdot e^{0.05\cdot 1}$

≈ 105.13

$t = 30$ (years)

$B(30) = 100\cdot e^{0.05\cdot 30}$

≈ 448.15

Idea: The longer we keep the money in, the faster the amount grows.

Formulas for compound interest: summary

Given a principal investment P and an annual interest rate r, if

- the interest is compounded k times, then the balance at the end of the year is
$$B = P\left(1 + \frac{r}{k}\right)^k.$$
- the interest is compounded k times, then the balance at the end of t years is
$$B(t) = P\left(1 + \frac{r}{k}\right)^{kt}.$$
- the interest is compounded continuously, then the balance at the end of t years is
$$B(t) = Pe^{rt}.$$

Finding present values

Question. How much money should you invest today to get a certain amount at the end of a time period?
Rephrase. Given the balance $B(t)$ at time t (future value), find the principal P needed to invest today (present value).

2 cases : $\boxed{1}$ $B(t) = P\left(1+\frac{r}{k}\right)^{kt}$, $P = \dfrac{B(t)}{\left(1+\frac{r}{k}\right)^{kt}}$

$\boxed{2}$ $B(t) = Pe^{rt}$, then $P = \dfrac{B(t)}{e^{rt}}$

Example. You need \$5000 for a trip in 4 years. The annual interest rate is 6%. How much should you invest now if the interest rate is compounded continuously?

$5000 = B(t)$ $t = 4$

$r = 0.06$ continous $\Rightarrow$ case $\boxed{2}$

So $P = \dfrac{5000}{e^{0.06 \cdot 4}} = \dfrac{5000}{e^{0.24}}$

≈ 3933.14

Comparing investments

Question. Which investment is better: one with an annual interest rate of 6% compounded semiannually, or one with 5.9% rate compounded continuously?

(One) idea. Find the **effective rate** r_e, which is the simple annual rate equivalent to each investment. The higher effective rate wins!

Simple interest = $k=1$

2 cases:

(1) $B = P(1+r_e)$

Solve for r_e:

$$\cancel{P}(1+r_e) = \cancel{P}\left(1+\frac{r}{k}\right)^k$$

$$\boxed{r_e = \left(1+\frac{r}{k}\right)^k - 1}$$

e.g: $r = 6\% = 0.06$, $k=2$

$$r_e = \left(1+\frac{0.06}{2}\right)^2 - 1 = 0.0609$$

(2) Continuous:

$$\cancel{P}(1+r_e) = \cancel{P}e^r$$

$$\boxed{r_e = e^r - 1}$$

e.g: $r = 5.9\% = 0.059$

$$r_e = e^{0.059} - 1 \approx 0.0608$$

Since $0.0609 > 0.0608$,
Better choose 1st option.

Take a minute to digest!

Take a minute to write as much as you can remember from the past 20 minutes.

Logarithmic functions: introduction

Question. Suppose you invest \$4000 at the annual rate of 5% compounded continuously. How many years does it take for this money to double?

Idea. Start with formula: in this case, the balance after t year is

$$B(t) = 4000e^{0.0\cancel{0}5t}.$$

Since we want it to double, we want

$$2 \cdot \cancel{4000} = \cancel{4000}e^{0.0\cancel{0}5t}.$$

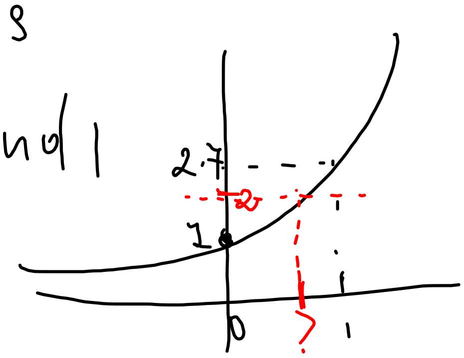
} solve for t

Solve for t: $e^{0.05t} = 2$

remember that $e^0 = 1$

$e^1 = 2.71\ldots > 2$

So $0.05t$ is between 0 and 1



$0 < 0.05t < 1$ [mult by 20

$0 < t < 20$ | $20 \cdot 0 < \underbrace{20 \cdot 0.05}_{1}t < 20 \cdot 1$

To get more precise answer, logarithms!

Logarithmic functions

Definition. If $a, b > 0$, $b \neq 1$, the **logarithm** of a in base b, written $\log_b a$ (or $\log_b(a)$, is the real number v such that $b^v = a$. .

$$\boxed{v = \log_b a \iff b^v = a}$$

logarithm is a hidden exponent

Tip: think of the logarithm as an exponent.

e.g: $\log_5(25) = ?$ then $5^{?} = 25$

since $5^2 = 25$, $\log_5 25 = 2$

If the base $b = e \approx 2.71828\ldots$, write $\log_e a = \ln(a)$ the "natural logarithm".

$$\boxed{v = \ln a \iff e^v = a}$$