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Applications of Logarithmic Functions

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Lecture 5

Logarithm rules (continued)

Let $u, v > 0, b > 0, b \neq 1$.

- Equality:
$$\log_b u = \log_b v \iff u = v.$$
- Inversion:
$$\log_b(b^u) = u \qquad \text{and} \qquad b^{\log_b u} = u.$$

Some examples of inversion: $7^x = 8$.
Taking $\log_7$ on both sides, we get

$$\log_7(7^x) = \log_7 8$$

The first inversion rule then gives

$$x = \log_7 8 \approx 1.06862\ldots.$$

Another example: $\log_5 x = \frac{1}{3}$.

second inversion: raise 5 to both sides

$$5^{\log_5 x} = 5^{\frac{1}{3}}$$
$$\parallel$$
$$x = 5^{1/3}$$

Solving logarithmic equations (cont.)

Use inversion rule 1: because we start with a power

Exercise 2. Solve for x the equation

$$3^{5x-1} = 12.$$

12 is not 3 power

Take $\log_3$ of both sides

$$\log_3(3^{5x-1}) = \log_3 12$$

$$5x - 1 = \log_3(12)$$

$$5x = \log_3(12) + 1$$

$$x = \frac{\log_3(12)+1}{5} \approx 0.652372$$

Solving logarithmic equations (cont.)

Inversion rule 2 : $b^{\log_b u} = u$

Exercise 3. Solve for x the equation

$$\log_2(2x - 20) = 4.$$

raise 2 to both sides

$$2^{\log_2(2x-20)} = 2^4$$

$$2x - 20 = 16$$

$$2x = 16 + 20$$

$$2x = 36$$

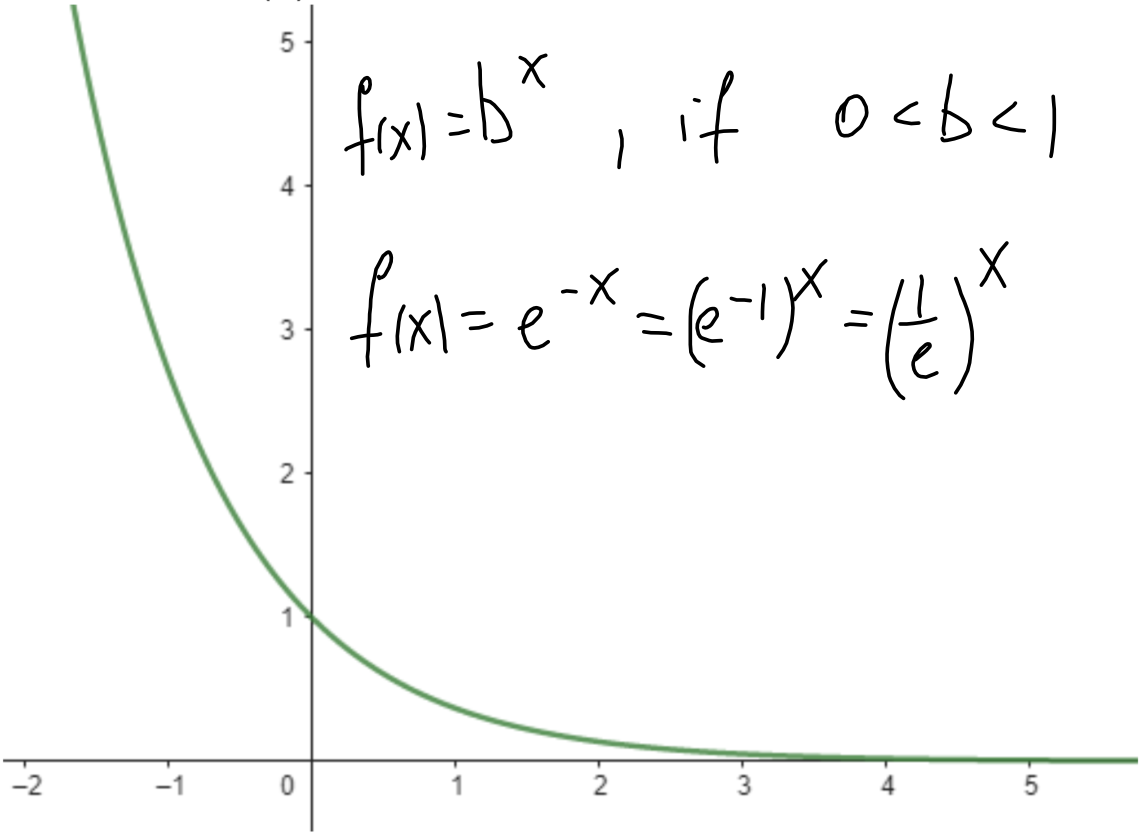
$$\boxed{x = 18}$$

Take a minute to digest!

Take a minute to write as much as you can remember from the past 15 minutes.

Applications for exponential decay

The graph of $f(x) = e^{-x}$ has exponential decay:



$f(x) = b^x \,, \text{ if } \; 0 < b < 1$

$f(x) = e^{-x} = (e^{-1})^x = \left(\frac{1}{e}\right)^x$

Application: radioactive decay

A radioactive sample of initial size Q_0 grams will decay to

$$\boxed{Q(t) = Q_0 e^{-kt}} \quad \text{grams in } t \text{ years.}$$

The **half-life** of the substance is the amount of time needed for half of the substance to decay:

$$\frac{1}{2} Q_0 = Q_0 e^{-kt}$$ solve for t

Given a specific k, we can solve for the half-life t.

Example. The amount of a certain radioactive substance after t years is

$$Q(t) = Q_0 e^{-0.003t}.$$

Find the half-life.

Solution: Solve $\frac{1}{2} \cancel{Q_0} = \cancel{Q_0} e^{-0.003t}$ for t

$$e^{-0.003t} = \frac{1}{2}$$

take $\ln = \log_e$ on both sides

$$\ln\left(e^{-0.003t}\right) = \ln \frac{1}{2}$$

$$-0.003t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{\ln\left(\frac{1}{2}\right)}{-0.003} \approx 231.05 \text{ (years)}$$

Application: carbon dating

^{14}C

Example. A fossil has 35% carbon 14 compared to the living sample. If the half-life of carbon 14 is 5730 years, how old is the fossil?

Idea for solution. The amount of carbon 14 in the sample after t years is

$$Q(t) = Q_0 e^{-kt}. \quad \left\} \begin{array}{l} \text{have} \\ \frac{1}{2}Q_0 = Q_0 e^{-k \cdot 5730} \end{array} \right.$$

We do not have k (yet), but we do have the half-life. So, we can find k, then solve for t given that at the present time, the amount of carbon 14 found in the sample is $\frac{35}{100}Q_0$.

To find the age, if we have k, we can solve for t

$$\frac{35}{100} Q_0 = Q_0 e^{-k \cdot t}$$

Solution: Half-life of ^{14}C is 5730 years,

$$\frac{1}{2}\cancel{Q_0} = \cancel{Q_0} e^{-k \cdot 5730} \quad \Big\} \text{ find } k.$$

$$e^{-5730k} = \frac{1}{2}$$

take ln of both sides

$$\ln(e^{-5730k}) = \ln\left(\tfrac{1}{2}\right)$$

$$-5730k = \ln\left(\tfrac{1}{2}\right)$$

$$k = \frac{\ln(\frac{1}{2})}{-5730} = -\frac{\ln(\frac{1}{2})}{5730} \quad \Big\} \text{ have } k.$$

Step 2: solve for t.

$$\frac{35}{100} Q_0 = Q_0 e^{-\left(-\frac{\ln(\frac{1}{2})}{5730} t\right)}$$

$$\frac{35}{100} \cancel{Q_0} = \cancel{Q_0} e^{\frac{\ln(\frac{1}{2})t}{5730}}$$

take ln of both sides.

$$\ln\left(\frac{35}{100}\right) = \ln\left(e^{\frac{\ln(\frac{1}{2})t}{5730}}\right)$$

inversion rule

$$\ln\left(\frac{35}{100}\right) = \frac{\ln(\frac{1}{2})t}{5730}$$

$$5730 \cdot \ln\left(\frac{35}{100}\right) = \ln\left(\tfrac{1}{2}\right)t$$

$$t = \frac{\ln(\frac{35}{100}) \cdot 5730}{\ln(\frac{1}{2})} \approx 8678.5 \text{ years.}$$

Take a minute to digest!

Take a minute to write as much as you can remember from the past 15 minutes.

Recall differentiation rules : $f'(x) = (f(x))'$

- Power: $(x^a)' = ax^{a-1}$

  e.g: $(x^5)' = 5x^{5-1} = 5x^4$

- Product: $[f(x)g(x)]' = g(x)f'(x) + g'(x)f(x)$

  Example:

$$[(x^2+2)(x^2-3)]' = (x^2+2)' \cdot (x^2-3)$$

$$+ (x^2-3)' \cdot (x^2+2) = 2x(x^2-3) + 2x(x^2+2)$$

- Quotient: $\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ $\quad = 4x^3 - 2x$

  Example:

$$\left[\frac{x^3-5}{x^7+x}\right]' = \frac{(x^3-5)' \cdot (x^7+x) - (x^7+x)' \cdot (x^3-5)}{(x^7+x)^2}$$

$$= \frac{3x^2(x^7+x) - (7x^6+1) \cdot (x^3-5)}{(x^7+x)^2}$$