McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

Differentiation of logarithmic and exponential functions

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Lecture 6

Log base change formula

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Example:

$$\log_7 8 = \frac{\ln 8}{\ln 7} \approx 1.06862\ldots$$

Differentiation of exponential and log functions

Exponential:

$$(b^x)' = b^x \ln b \xrightarrow[b=e]{\ln(e)=1} (e^x)' = e^x$$

Logarithmic:

$$(\log_b x)' = \frac{1}{x \ln b} \xrightarrow[\ln e = 1]{b=e} (\ln x)' = \frac{1}{x}$$

Examples:

$$\left[\frac{x^3 - x - 1}{3^x}\right]' = \frac{3^x \cdot (x^3-x-1)' - (x^3-x-1)\cdot(3^x)'}{(3^x)^2}$$

$\ln 3$

$$\boxed{\left(\frac{HI}{LO}\right)' = \frac{LO \cdot dHI - HI \cdot dLO}{LO \cdot LO}}$$

$$= \frac{\cancel{3^x}(3x^2-1) - (x^3-x-1)\cdot \cancel{3^x}\ln 3}{\cancel{3^x}\cdot 3^x}$$

$$= \frac{3x^2-1 - (x^3-x-1)\ln 3}{3^x}$$

$$[(x^2+7)\ln x]' = (x^2+7)\cdot[\ln x]' + \ln x \cdot (x^2+7)'$$

$$\begin{array}{ccc} x^2+7 & & 2x \\ \ln x & & \frac{1}{x} \end{array}$$

$$= (x^2+7)\cdot \frac{1}{x} + (\ln x)\cdot 2x$$

$$= \frac{x^2+7}{x} + 2x \ln x$$

The chain rule

$$[f(g(x))]' = g'(x)f'(g(x))$$

$$x \xrightarrow{g'(x)} g(x) \xrightarrow{f'(g(x))} f(g(x))$$

Examples:

$$[(x^2+2x+3)^{25}]' =$$

$$(\sqrt{x^3-x})' =$$

$$x \xrightarrow{\text{derivative is } 2x+2+0} x^2+2x+3 \xrightarrow{\text{derivative is } 25(x^2+2x+3)^{24}} (x^2+2x+3)^{25}$$

big derivative

$$(2x+2)\cdot 25(x^2+2x+3)^{24}$$

$$\parallel$$

$$[(x^2+2x+3)^{25}]'$$

$$\left(\sqrt{x^3-x}\right)' = \left((x^3-x)^{1/2}\right)'$$

$$x \xrightarrow{3x^2-1} x^3-x \xrightarrow{\frac{1}{2}(x^3-x)^{-1/2}} (x^3-x)^{1/2}$$

$$\left(\sqrt{x^3-x}\right)' = (3x^2-1)\cdot\frac{1}{2}(x^3-x)^{-1/2}$$

$$= \frac{3x^2-1}{2(x^3-x)^{1/2}} = \frac{3x^2-1}{2\sqrt{x^3-x}}$$

The chain rule for $e^{u(x)}$

$$\boxed{(e^x)' = e^x}$$

$$[f(g(x))]' = g'(x)f'(g(x))$$

$$x \xrightarrow[u'(x)]{} u(x) \xrightarrow[e^{u(x)}]{} e^{u(x)}$$

gives

$$\boxed{(e^{u(x)})' = u'(x)e^{u(x)}}$$

Example:

$$(e^{x^2+2x+1})' = (x^2+2x+1)' \cdot e^{x^2+2x+1}$$

$$= (2x+2+0)\, e^{x^2+2x+1}$$

The chain rule for $\ln(u(x))$

$$(\ln x)' = \frac{1}{x}$$

$$[f(g(x))]' = g'(x)f'(g(x))$$

$$x \xrightarrow{\quad} u(x) \xrightarrow{\quad} \ln(u(x))$$

gives $u'(x)$ $\qquad \frac{1}{u(x)}$

$$\boxed{[\ln(u(x))]' = u'(x)\frac{1}{u(x)}}$$

Example:

$$[\ln(x^3 + 4x + 2)]' = (x^3+4x+2)' \cdot \frac{1}{x^3+4x+2}$$

$$= \frac{3x^2+4}{x^3+4x+2}$$

Exercises

Exercise 1. Differentiate the function

$$f(x) = (x^2 + 4x + 6)e^{6x}$$

We use product + chain rules

Product:

$x^2+4x+6 \qquad 2x+4$

$e^{6x} \qquad (e^{6x})' = (6x)' \cdot e^{6x}$

Chain

$= 6e^{6x}$

$\boxed{[e^{u(x)}]' = u'(x)e^{u(x)}}$

$f'(x) = (x^2+4x+6) \cdot 6e^{6x} + e^{6x} \cdot (2x+4)$

$= e^{6x}\left(6(x^2+4x+6) + (2x+4)\right)$

$= e^{6x}(6x^2+24x+36+2x+4)$

$= e^{6x}(6x^2+26x+40)$

Exercises

Exercise 2. Differentiate the function

$$f(x) = (\ln(2x+5))^4$$

$$x \xrightarrow[2]{} 2x+5 \xrightarrow[\frac{1}{2x+5}]{} \ln(2x+5) \xrightarrow[4[\ln(2x+5)]^3]{} [\ln(2x+5)]^4$$

So $f'(x) = \dfrac{2 \cdot 4[\ln(2x+5)]^3}{2x+5}$

$$= \frac{8[\ln(2x+5)]^3}{2x+5}$$

Exercises

Exercise 3. Differentiate the function

$$f(x) = \frac{x^3}{e^x + 2}$$

quotient rule

$$f'(x) = \frac{(e^x+2)\cdot(x^3)' - x^3\cdot(e^x+2)'}{(e^x+2)^2}$$

$(e^x)' = e^x$

$$= \frac{(e^x+2)\cdot(3x^2) - x^3\cdot e^x}{(e^x+2)^2}$$