



FACULTY OF SCIENCE
Department Of Mathematics & Statistics

Differentiation of logarithmic and exponential functions

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Lecture 6



Log base change formula

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Example:

$$\log_7 8 = \frac{\ln 8}{\ln 7} \approx 1.06862\dots$$



Differentiation of exponential and log functions

Exponential:

$$(b^x)' = b^x \ln b \quad \begin{array}{c} \ln(e) = 1 \\ \xrightarrow{b=e} \\ b=e \end{array} \quad (e^x)' = e^x$$

Logarithmic:

$$(\log_b x)' = \frac{1}{x \ln b} \quad \begin{array}{c} b=e \\ \xrightarrow{\ln e = 1} \\ \ln e = 1 \end{array} \quad (\ln x)' = \frac{1}{x}$$

Examples:

$$\left[\frac{x^3 - x - 1}{3^x} \right]' = \frac{3^x \cdot (x^3 - x - 1)' - (x^3 - x - 1) \cdot (3^x)'}{(3^x)^2} \quad \ln 3$$

$$\left(\frac{HI}{LO} \right)' = \frac{LO \cdot dHI - HI \cdot dLO}{LO \cdot LO} = \frac{\cancel{3^x} (3x^2 - 1) - (x^3 - x - 1) \cdot \cancel{3^x} \ln 3}{3^x \cdot 3^x} = \frac{3x^2 - 1 - (x^3 - x - 1) \ln 3}{3^x}$$

$$[(x^2 + 7) \ln x]' = (x^2 + 7) \cdot (\ln x)' + \ln x \cdot (x^2 + 7)'$$

$$\begin{array}{ccc} x^2 + 7 & \nearrow & 2x \\ \ln x & \searrow & \frac{1}{x} \end{array}$$

$$= (x^2 + 7) \cdot \frac{1}{x} + (\ln x) \cdot 2x$$

$$= \frac{x^2 + 7}{x} + 2x \ln x$$

The chain rule

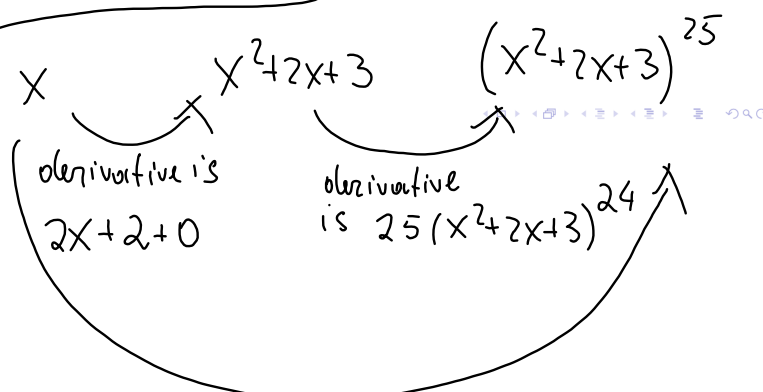
$$[f(g(x))]' = g'(x)f'(g(x))$$



$$[(x^2 + 2x + 3)^{25}]' =$$

...

$$(\sqrt{x^3 - x})' =$$

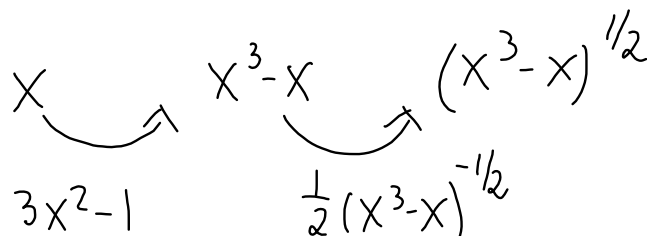


big derivative

$$(2x + 2) \cdot 25(x^2 + 2x + 3)^{24}$$

$$[(x^2 + 2x + 3)^{25}]'$$

$$(\sqrt{x^3 - x})' = ((x^3 - x)^{1/2})'$$

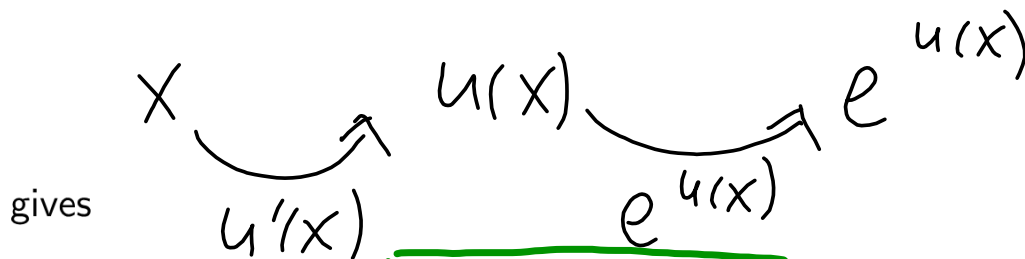


$$\begin{aligned} (\sqrt{x^3 - x})' &= (3x^2 - 1) \cdot \frac{1}{2}(x^3 - x)^{-1/2} \\ &= \frac{3x^2 - 1}{2(x^3 - x)^{1/2}} = \frac{3x^2 - 1}{2\sqrt{x^3 - x}} \end{aligned}$$

The chain rule for $e^{u(x)}$

$$[f(g(x))]' = g'(x)f'(g(x))$$

$$(e^x)' = e^x$$



$$(e^{u(x)})' = u'(x)e^{u(x)}$$

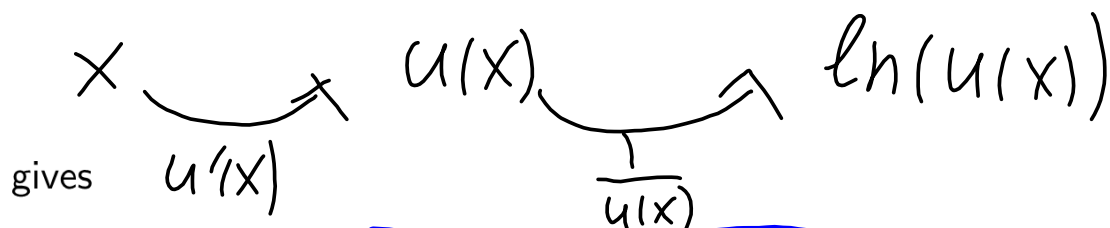
Example:

$$\begin{aligned} (e^{x^2+2x+1})' &= (x^2+2x+1)' \cdot e^{x^2+2x+1} \\ &= (2x+2+0) e^{x^2+2x+1} \end{aligned}$$

The chain rule for $\ln(u(x))$

$$[f(g(x))]' = g'(x)f'(g(x))$$

$$(\ln x)' = \frac{1}{x}$$



$$[\ln(u(x))]' = u'(x) \frac{1}{u(x)}$$

Example:

$$[\ln(x^3 + 4x + 2)]' = (x^3 + 4x + 2)' \cdot \frac{1}{x^3 + 4x + 2}$$

$$= \frac{3x^2 + 4}{x^3 + 4x + 2}$$



Exercises

Exercise 1. Differentiate the function

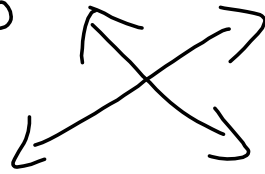
$$f(x) = (x^2 + 4x + 6)e^{6x}$$

We use product + chain rules

Product:

$$x^2 + 4x + 6$$

$$e^{6x}$$



$$2x + 4$$

chain

$$(e^{6x})' = (6x)' \cdot e^{6x}$$

$$= 6e^{6x}$$

$$[e^{u(x)}]' = u'(x)e^{u(x)}$$

$$f'(x) = (x^2 + 4x + 6) \cdot 6e^{6x} + e^{6x} \cdot (2x + 4)$$

$$= e^{6x} (6(x^2 + 4x + 6) + (2x + 4))$$

$$= e^{6x} (6x^2 + 24x + 36 + 2x + 4)$$

$$= e^{6x} (6x^2 + 26x + 40)$$

Exercises

Exercise 2. Differentiate the function

$$f(x) = (\ln(2x + 5))^4$$

$$\begin{array}{ccccccc} x & \xrightarrow{\quad} & 2x+5 & \xrightarrow{\quad} & \ln(2x+5) & \xrightarrow{\quad} & [\ln(2x+5)]^4 \\ & \underbrace{\quad} & & \underbrace{\quad} & & \underbrace{\quad} & \\ & 2 & & \frac{1}{2x+5} & & 4[\ln(2x+5)]^3 & \end{array}$$

$$\text{So } f'(x) = \frac{2 \cdot 4 [\ln(2x+5)]^3}{2x+5}$$

$$= \frac{8 [\ln(2x+5)]^3}{2x+5}$$

Exercises

Exercise 3. Differentiate the function

$$f(x) = \frac{x^3}{e^x + 2} \quad \text{by quotient rule}$$

$$\begin{aligned} f'(x) &= \frac{(e^x + 2) \cdot (x^3)' - x^3 \cdot (e^x + 2)'}{(e^x + 2)^2} \\ &= \frac{(e^x + 2) \cdot (3x^2) - x^3 \cdot e^x}{(e^x + 2)^2} \end{aligned}$$

$(e^x)' = e^x$