FACULTY OF SCIENCE
Department Of Mathematics & Statistics

Section 5.1. Solving differential equations

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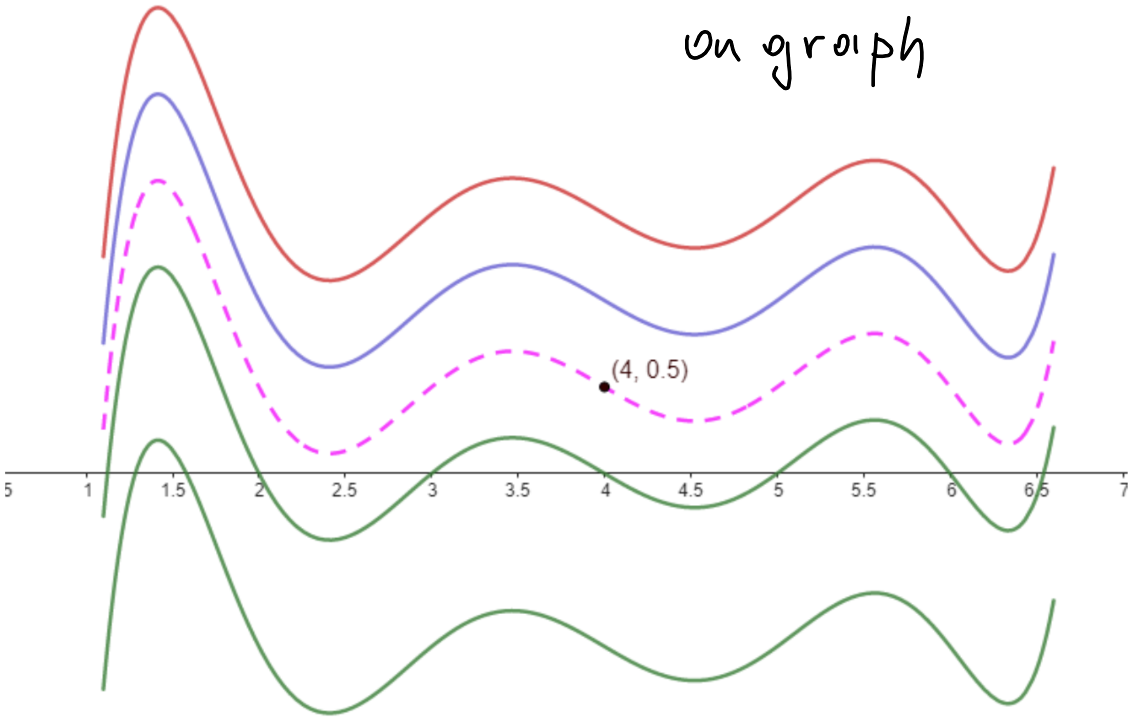
Lecture 14

Finding specific antiderivatives; $\underbrace{\text{Initial Value Problems}}_{\text{have one value on graph}}$



Example

Solve the following initial value problem:

$$\frac{dy}{dx} = e^{-x}, \qquad \text{where } y = 3 \;\text{ when }\; x = 0.$$

Idea. Recall that if $y = f(x)$, then $\frac{dy}{dx} = f'(x)$. To find the function $f(x)$, we integrate both sides.

Solution.

Step 1: $f(x) = \int e^{-x}\,dx = -e^{-x} + C$

Step 2: Solve for C if $y = 3$ and $x = 0$

$3 = -e^{-0} + C = -1 + C$ so $C = 4$

Answer: $f(x) = -e^{-x} + 4$

Additional practice problems for solving initial value problems:
Ex. 31, 33, 35, 37, 39, 41 in Chapter 5.1

Word problems I

The marginal revenue derived from producing q units of a certain commodity is $R'(q) = 4q - 1.2q^2$ dollars ~~per unit~~. If the revenue derived from producing 20 units is \$30,000, how much revenue should be expected from producing 40 units?

Solution. Step 1: Have $R'(q)$, can find $R(q)$ by integrating.

$$R(q) = \int (4q - 1.2q^2)\,dq = \frac{4}{2}q^2 - \frac{1.2}{3}q^3 + C$$
$$= 2q^2 - 0.4q^3 + C$$

Step 2: Solve for the specific constant C, knowing that $R(20) = 30,000$.

$$2 \cdot 20^2 - 0.4 \cdot 20^3 + C = 30000$$
$$800 - 3200 + C = 30000$$
$$C = 30000 + 3200 - 800 = 32400.$$

So $R(q) = 2q^2 - 0.4q^3 + 32400$

<u>Step 3</u>: Evaluate $R(40) = 2 \cdot 40^2 - 0.4 \cdot 40^3 + 32400$
$$= 10000$$

Take a minute to digest!

Take a minute to jot down what you can remember from these past 2 examples. For example, in your own words, what is an initial value problem?

Word problems II

A car travelling at 67 ft/sec decelerates at the constant rate of 23 ft/sec^2 when the brakes are applied.

(a) Find the velocity $v(t)$ of the car t seconds after the brakes are applied. $t=0$: initial point in time

Solution.

Distance	Velocity	Acceleration
$s(t)$	$v(t)=s'(t)$	$a(t)=v'(t)$
$s(0)=0$	$v(0)=67$	$a(t)=-23$

(a) $v(t)=\int v'(t)\,dt=\int a(t)\,dt=\int(-23)\,dt$

$=-23t+C$

Solve for C: $v(0)=67$

$-23\cdot 0+C=67$ so $C=67$

$\boxed{v(t)=-23t+67.}$

Word problems II (part 2)

(b) Find the distance $s(t)$ from the point where the brakes are applied.

In (a), $v(t) = -23t + 67$.

$$s(t) = \int s'(t)\,dt = \int (-23t + 67)\,dt$$

$$= -\frac{23}{2}t^2 + 67t + \tilde{C}$$

Solve for $\tilde{C}$ if $s(0) = 0$

$-\frac{23}{2} \cdot 0^2 + 67 \cdot 0 + \tilde{C} = 0$ so $\tilde{C} = 0$

So $\boxed{s(t) = -\frac{23}{2}t^2 + 67t}$

Word problems II (part 3)

(c) How far does the car travel before coming to a full stop?

Solution: The car stops when velocity is 0. Solve for t $v(t)=0$:

$$-23t+67=0$$

$$t=\frac{-67}{-23}=\frac{67}{23}$$ (this is when the car stops)

To find distance, $s\left(\frac{67}{23}\right)=-\frac{23}{2}\cdot\left(\frac{67}{23}\right)^2+67\cdot\left(\frac{67}{23}\right)$

$$=\frac{4489}{46}\approx 97.59\,(ft)$$

Additional word problem practice:
Ex. 47, 49, 53, 55, 61, 63, 65 in Chapter 5.1

Differential equations

So far, we have been solving what are called "differential equations": given the derivative, find the function.

$$\frac{dy}{dx} = g(x)$$ **Method to solve:** integrate the right-hand side

Now, we look at a specific type of differential equation, called **separable**:

$$\frac{dy}{dx} = \frac{h(x)}{g(y)}, \quad \frac{dy}{dx} = h(x)g(y), \quad \text{or} \quad \frac{dy}{dx} = \frac{g(y)}{h(x)}.$$

Ex: $\frac{dy}{dx} = \frac{x^2+x+2}{y^2+9y}$ $\qquad \frac{dy}{dx} = \ln(2x^2+1)e^y$

$$\frac{dy}{dx} = \frac{\ln(2y-5)}{x^2+5x+1}$$

Nonexample: $\cancel{\frac{dy}{dx} = x^2+y^2}$