FACULTY OF SCIENCE
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Section 5.1. Solving separable differential equations

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Lecture 15

Separable differential equations

$\frac{dy}{dx} = f'(x)$, $y = f(x)$

Separable equations are ofthe form

$$\frac{dy}{dx} = \frac{h(x)}{g(y)}, \quad \frac{dy}{dx} = h(x)g(y), \quad \text{or} \quad \frac{dy}{dx} = \frac{g(y)}{h(x)}.$$

Examples.

$$\frac{dy}{dx} = \frac{x^2+x+2}{y^2+9y}, \quad \frac{dy}{dx} = \ln(x^2+2)e^y, \quad \frac{dy}{dx} = \frac{\ln(2y-5)}{x^2+5x+1}$$

Non-example. $\cancel{\frac{dy}{dx} = 3x + 2y}$

Products and quotients ok, but do not mix up variables in sums.

Separable differential equations

$\frac{dy}{dx} = \text{limit} \frac{\text{change in } y}{\text{change in } x}$

These equations are separable in the sense that we can separate expressions involving x and y on two sides of the equation.

$$\frac{dy}{dx} = \frac{h(x)}{g(y)} \leadsto \int g(y)\,dy = \int h(x)\,dx$$

$$\frac{dy}{dx} = h(x)g(y) \leadsto \int \frac{1}{g(y)}\,dy = \int h(x)\,dx$$

$$\frac{dy}{dx} = \frac{g(y)}{h(x)} \leadsto \int \frac{1}{g(y)}\,dy = \int \frac{1}{h(x)}\,dx$$

Method to solve: now integrate both sides.
For example,

$$\int g(y)dy = \int h(x)dx$$

Example

Find a general solution for

$$\frac{dy}{dx} = \frac{x}{3y^2}$$

Solution.

Step 1: Separate and integrate

$$\int 3y^2\,dy = \int x\,dx$$

$$\frac{\cancel{3}}{\cancel{3}} y^3 + C_1 = \frac{1}{2}x^2 + C_2$$

$$y^3 = \frac{1}{2}x^2 + C \qquad (C_2 - C_1 = C)$$

$$y = \sqrt[3]{\frac{1}{2}x^2 + C}$$

Example

Solve

$$\frac{dy}{dx} = xy, \quad \text{if} \quad y = 2 \text{ at } x = 1$$

Solution.

Separate and integrate

$$\int \frac{1}{y}\, dy = \int x \, dx$$

$$\ln(|y|) = \frac{1}{2}x^2 + C$$

Trick: Solve for C immediately after integrating.

$$\ln(|2|) = \frac{1}{2}\cdot 1^2 + C \quad \text{so } C = \ln(2) - \frac{1}{2}$$

Example (cont.)

$$\ln(|y|) = \frac{1}{2}x^2 + \ln(2) - \frac{1}{2}$$

Take e to both sides

$$e^{\ln(|y|)} = e^{\frac{1}{2}x^2 + \ln(2) - \frac{1}{2}}$$

$$|y| = \frac{e^{\frac{1}{2}x^2} \cdot e^{\ln(2)}}{e^{1/2}} = \frac{2e^{\frac{1}{2}x^2}}{\sqrt{e}}$$

Trick 2: If $|x|$ or $|y|$, write $|x| = x$ or $|y| = y$ if in initial value $x > 0$ or $y > 0$. If $x < 0$ or $y < 0$, $|x| = -x$ or $|y| = -y$.

Here, $|y| = y$ b/c $y = 2$ in initial value. (Aside: if $y = -2$, then $|y| = -y$)

Here, $\boxed{y = \frac{2e^{1/2 x^2}}{\sqrt{e}}}$ Answer

Then at $x = 1$, $y = \frac{2e^{1/2 \cdot 1^2}}{\sqrt{e}} = \frac{2\cancel{\sqrt{e}}}{\cancel{\sqrt{e}}} = 2.$

Example

$$= \frac{\sqrt{y}}{\sqrt{x}}$$

Solve

$$\frac{dy}{dx} = \sqrt{\frac{y}{x}}, \quad \text{if} \quad y = 1 \ \text{at} \ x = 1$$

Solution.

$$\int \frac{1}{\sqrt{y}}\,dy = \int \frac{1}{\sqrt{x}}\,dx$$

$$\int y^{-1/2}\,dy = \int x^{-1/2}\,dx$$

$$\frac{1}{-\frac{1}{2}+1}\,y^{1/2} = \frac{1}{-\frac{1}{2}+1}\,x^{1/2} + C$$

$$2y^{1/2} = 2x^{1/2} + C$$

Example (cont.)

$2y^{1/2} = 2x^{1/2} + C$; $y = 1$ at $x = 1$

Solve for C: $\cancel{2 \cdot 1^{1/2}} = \cancel{2 \cdot 1^{1/2}} + C$

so $C = 0$.

Then $\cancel{2} y^{1/2} = \cancel{2} x^{1/2}$. Square both sides

$(y^{1/2})^2 = (x^{1/2})^2$

$\boxed{y = x}$

Take a minute to digest!

Take a minute to jot down what you can remember from these past 2 examples. For example, what does a separable equation look like? And what are the steps for solving one?

Word problem I

Separable equations show up naturally in real-world applications.

Ex. A fruit fly population grows at a rate of 25% per day. If a biologist has 10 fruit flies today, how many will they have in 7 day?

Solution. For word problems, first condense the information and translate problem statement into a mathematical equation.

Soln: $P(t)$ = population at time t

$$P'(t) = \frac{dP}{dt} = 0.25\, P = \frac{1}{4} P$$

$$\frac{dP}{dt} = \frac{1}{4} P$$

Initial value: today $t=0$ has $P(0) = 10$

Find $P(7)$.

Word problem I (cont.)

$\frac{dP}{dt} = \frac{1}{4} P$ } separable : $\int \frac{1}{P}\, dP = \int \frac{1}{4}\, dt$

$\ln(|P|) = \frac{1}{4}t + C$. Solve for C:

$\ln(10) = \frac{1}{4} \cdot 0 + C$ so $C = \ln(10)$

Therefore, $\ln(|P|) = \frac{1}{4}t + \ln(10)$

Because $P > 0$, $|P| = P$

$\ln(P) = \frac{1}{4}t + \ln(10)$.