McMaster University

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Section 5.1. Solving separable differential equations (cont).
Section 5.2: Integration by substitution.

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Lecture 16

Word problem I (cont).

Ex. A fruit fly population grows at a rate of 25% per day. If a biologist has 10 fruit flies today, how many will they have in 7 day?

Solution (cont). Last time, we got $\ln(P) = \frac{1}{4}t + \ln(10)$.

$$\left[\text{Recall: } \frac{dP}{dt} = \frac{1}{4}P \rightsquigarrow \int \frac{1}{P}\,dP = \int \frac{1}{4}\,dt\right]$$

$$\ln(P) = \frac{1}{4}t + \ln(10)$$

$$e^{\ln(P)} = e^{\frac{1}{4}t + \ln(10)}$$

$$P = e^{\frac{1}{4}t} \cdot e^{\ln(10)} = 10e^{\frac{1}{4}t}$$

In 7 days, have $P(7) = 10e^{\frac{1}{4}\cdot 7}$

$$\approx 57.55$$

Answer: Between 57 and 58 flies.

Word problem II

Ex. Suppose your investment grow (continuously) at a yearly rate of 5%. Write an equation describing the balance as a function of time t (in years).

Solution.

$$\frac{dB}{dt} = 0.05\,B \quad \text{(rate of growth of balance)}$$

separate, integrate

$$\int \frac{1}{B}\,dB = \int \frac{5}{100}\,dt$$

$$\ln(|B|) = \frac{5}{100}t + C \qquad \Big|\; B \geq 0 \text{ so } |B| = B$$

$$\ln(B) = \frac{5}{100}t + C$$

$$e^{\ln(B)} = e^{\frac{5}{100}t + C}$$

$$B = e^{\frac{5}{100}t} \cdot \underbrace{e^{C}}_{P}$$

So $\boxed{B = Pe^{\frac{5}{100}t}}$

Note: e^C is info about the initial value.

Here, the initial value is the principal investment P.

Additional practice problems for solving separable differential equations.

Ex. 43, 45 in Chapter 5.1, and some exercises from Assignment 5.

Section 5.2: Integration by substitution

Example. Suppose we need to integrate

$$\int (4x^3 + 3x^2)e^{x^4+x^3}\,dx.$$

Issue: Not one of the basic integration formulas, and it doesn't look like we can immediately make it into one.
Idea: Examine the factor $4x^3 + 3x^2$ and the exponent $x^4 + x^3$ more closely.

Notice: $(x^4+x^3)' = 4x^3+3x^2$

Let $u = x^4+x^3$

Then $u'(x) = \frac{du}{dx} = 4x^3+3x^2$

Separate the variables:

$$du = (4x^3+3x^2)\,dx$$

Example (cont.)

Need: $\int e^{\boxed{x^4+x^3}} \cdot \boxed{(4x^3+3x^2)\,dx} = ?$

$u = \boxed{x^4+x^3}$ $\qquad du = \boxed{(4x^3+3x^2)\,dx}$

Substitute u in the integral:

$$\int e^u \, du = e^u + C \quad \left(\text{one of the original formulas!}\right)$$

Substitute $u = x^4 + x^3$ back:

$$= e^{x^4+x^3} + C.$$

Check: $\left(e^{x^4+x^3} + C\right)'$

$$= (x^4+x^3)' \cdot e^{x^4+x^3} + 0$$

$$= (4x^3+3x^2) \cdot e^{x^4+x^3}$$

New integration technique: u-substitution

Generally speaking, we get

$$\int g(u(x))\boxed{u'(x)dx} = \int g(u)du$$

$$u'(x) = \frac{du}{dx} \quad \text{so} \quad du = \boxed{u'(x)dx}$$

This rule is the chain rule in reverse:

$$g(u(x)) + C = \int [g(u(x))]' = \int g'(u(x))u'(x).$$

Example

$$\int \boxed{\frac{2x-3}{\sqrt{x^2-3x}}dx} =$$

Solution.

Examine $2x-3$ and x^2-3x.

$(x^2-3x)' = 2x-3$

let $u = \boxed{x^2-3x}$ (choose u to be the more complicated one)

$\frac{du}{dx} = 2x-3 \rightsquigarrow du = \boxed{(2x-3)dx}$

Substitute: $\int \frac{du}{\sqrt{u}} = \int u^{-1/2}du = \frac{1}{-\frac{1}{2}+1}u^{-1/2+1} + C$

$= 2u^{1/2} + C$

Substitute back: $= 2(x^2-3x)^{1/2} + C$

Example

$$\int \frac{3x^2+5}{x^3+5x+1}dx =$$

Solution.

What is $u = ?$

Let $u = x^3 + 5x + 1$

$u'(x) = 3x^2 + 5$ ✓