McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

5.3 Definite integrals with u-substitution
5.4 Applications

Angelica Babei

MATH 1MM3 Winter 2023
Lecture 22

Examples

let $u = 2x + 6$

$$\int_{-3}^{0} (2x+6)^4 \, dx$$

$\rightarrow 2 \cdot 0 + 6 = 6$

$\rightarrow 2 \cdot (-3) + 6 = 0$

Soln. Last time we got that this equals

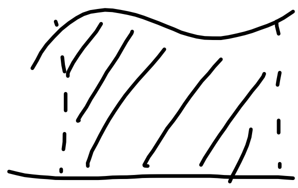
$$\int_{0}^{6} \frac{1}{2} u^4 \, du = \frac{1}{2} \cdot \frac{1}{5} u^5 \Big|_{0}^{6} = \frac{1}{10}(6^5 - 0^5)$$

$$= 777.6$$

Examples

Find the area of the region that lies under the graph of

$$y = \frac{1}{\sqrt{9-2x}} \geq 0$$



on the interval

$$-12 \leq x \leq 0$$

Soln:

$$\int_{-12}^{0} \frac{1}{\sqrt{9-2x}}\,dx = \int_{-12}^{0} (9-2x)^{-1/2}\,dx$$

$x = -12$ $\quad$ $x = 0$

let $\boxed{u = 9-2x} \rightsquigarrow du = -2\,dx$

$-\frac{1}{2}\,du = dx$

$x = 0 \rightsquigarrow \boxed{9 - 2\cdot 0} = 9$

$x = -12 \rightsquigarrow 9 - 2\cdot(-12) = 33$

$$= \int_{33}^{9} u^{-1/2} \cdot \left(-\frac{1}{2}\right) du$$

Example (cont.)

$$= -\frac{1}{2}\int_{33}^{9} u^{-1/2}\,du = -\frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}\,u^{-1/2+1}\Big|_{33}^{9}$$

$$= -u^{1/2}\Big|_{33}^{9} = -\left(\sqrt{9} - \sqrt{33}\right) = \sqrt{33} - 3$$

Example

The rate of production of a new type of cellphone is $1500(2 - te^{-t^2})$ units per month. How many were produced in the third month?

Soln. We know the rate of production

$$\boxed{P'(t)} = 1500(2 - te^{-t^2}) \quad \text{(units/month)}.$$

Need to evaluate: Bounds = # units in 3 months − # units in 2 months

$$\int_2^3 1500(2 - te^{-t^2})\,dt$$

$$= 1500 \int_2^3 (2 - te^{-t^2})\,dt = 1500\left[\int_2^3 2\,dt - \int_2^3 te^{-t^2}\,dt \right]$$

① ②

$$① \int_2^3 2\,dt = 2t\Big|_2^3 = 2(3-2) = 2$$

Example (cont.)

② $\int_2^3 t e^{-t^2}\,dt = \int_{-4}^{-9} e^u \cdot \left(-\frac{1}{2}\right) du$

Set $u = \boxed{-t^2}$

$\frac{du}{dt} = -2t \rightsquigarrow -\frac{1}{2}du = t\,dt$

$\int e^u \cdot \left(-\frac{1}{2}\right) du$

$t = 3 \rightsquigarrow \boxed{-3^2 = -9}$

$t = 2 \rightsquigarrow -2^2 = -4$

Example (cont.)

$$= -\frac{1}{2}\int_{-4}^{-9} e^u \, du = -\frac{1}{2} e^u \Big|_{-4}^{-9} = -\frac{1}{2}\left(e^{-9} - e^{-4}\right)$$

Put everything together:

$$1500 \cdot (1 - 2) = 1500\left(2 - \left(-\frac{1}{2}\right)\left(e^{-9} - e^{-4}\right)\right)$$

$$= 3000 + 750\left(e^{-9} - e^{-4}\right)$$

$$\approx 2986 \text{ (units)}$$

Example

A study indicates that t months from now the population of a certain town will be growing at the $\boxed{\text{rate}}$ of $(5 + 3t(1 + 5t)^{2/3})$ people per month. By how much will the population of the town increase over the next 8 months?

Soln. We know the rate of growth

$$\boxed{P'(t)} = 5 + 3t(1 + 5t)^{2/3} \quad \text{(people/month)}.$$

Need to evaluate:

$$\int_0^8 (5 + 3t(1+5t)^{2/3})\,dt$$

$$= \underbrace{\int_0^8 5\,dt}_{1} + 3\underbrace{\int_0^8 t(1+5t)^{2/3}\,dt}_{2}$$

1 $\displaystyle\int_0^8 5\,dt = 5t\Big|_0^8 = 5(8-0) = 40$

Example (cont.)

$$3\int_0^8 \boxed{t}(1+5t)^{2/3}\,dt$$

???

let $u = 1+5t \rightsquigarrow du = 5\,dt$

$\frac{1}{5}du = dt$

$t = \frac{u-1}{5}$

$$= 3\int \frac{u-1}{5}\, u^{2/3}\left(\frac{1}{5}du\right)$$

$t=8 \rightsquigarrow 1+5\cdot 8 = 41$

$t=0 \rightsquigarrow 1+5\cdot 0 = 1$

Example (cont.)

$$= 3\int_1^{41} \frac{u-1}{5}\, u^{2/3} \cdot \frac{1}{5}du = \frac{3}{25}\int_1^{41} (u-1)u^{2/3}du$$

$$= \frac{3}{25}\int_1^{41} \left(u^{5/3} - u^{2/3}\right) du$$

$$= \frac{3}{25}\left(\frac{1}{\frac{5}{3}+1}u^{5/3+1} - \frac{1}{\frac{2}{3}+1}u^{2/3+1}\right)\Bigg|_1^{41}$$