FACULTY OF SCIENCE
Department Of Mathematics & Statistics

5.4 Applications

Angelica Babei

MATH 1MM3 Winter 2023
Lecture 23

Example

A study indicates that t months from now the population of a certain town will be growing at the rate of $(5 + 3t(1+5t)^{2/3})$ people per month. By how much will the population of the town increase over the next 8 months?

Soln. Last time: evaluated

$$\int_0^8 (5 + 3t(1+5t)^{2/3})dt = \int_0^8 5dt + 3\int_0^8 t(1+5t)^{2/3}dt$$

Last time

$$= 40 + \frac{3}{25}\left(\frac{1}{\frac{5}{3}+1}u^{5/3+1} - \frac{1}{\frac{2}{3}+1}u^{2/3+1}\right)\Bigg|_1^{41}$$

$$= 40 + \frac{3}{25}\left(\frac{3}{8}u^{8/3} - \frac{3}{5}u^{5/3}\right)\Bigg|_1^{41}$$

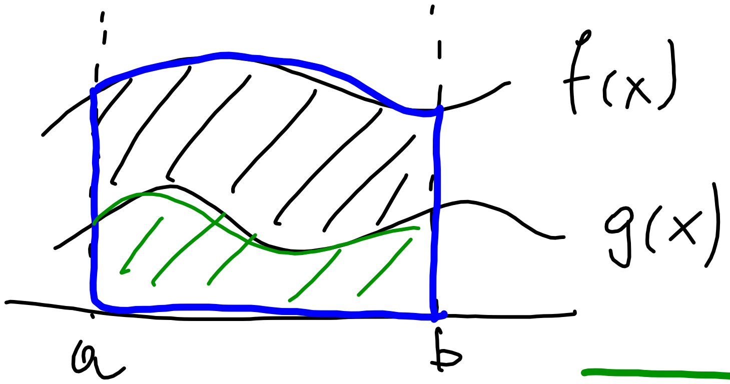
$$= 40 + \frac{3}{25}\left[\left(\frac{3}{8}41^{8/3} - \frac{3}{5}41^{5/3}\right) - \left(\frac{3}{8} - \frac{3}{5}\right)\right]$$

$$\approx 40 + \frac{3}{25}(7202.787 + 0.225) \approx 904$$

Additional practice problems for solving definite integrals
(including with u-substitution):
Ex. 33, 35, 37, 39, 41, 43, 53, 57, 63, 65, 67, 75, 77 in Section 5.3.

Section 5.4: Application 1, The area between curves

Let $f(x), g(x)$ have the following graphs. Can you think of a strategy on how to find the area between the two curves when $a \leq x \leq b$?

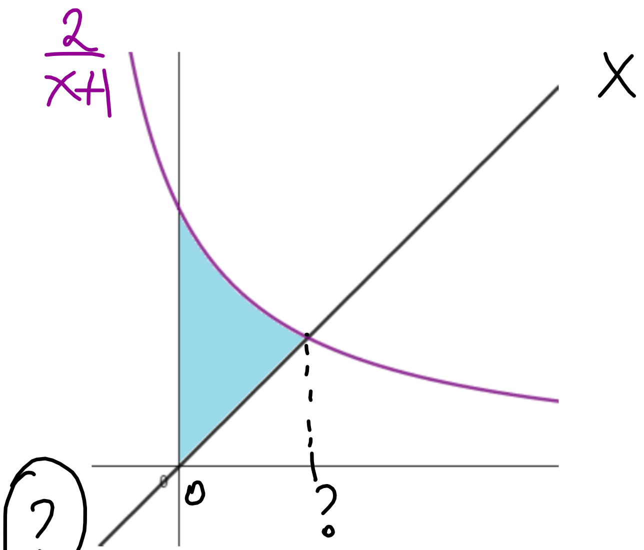


Area //// $= \boxed{\text{Area under } f(x)} - \boxed{\text{Area under } g(x)}$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx$$

Example

Find the shaded area, if the purple curve is $y = \frac{2}{x+1}$ and the line is $y = x$.



$$\text{Area} = \int_0^{?} \left(\frac{2}{x+1} - x \right) dx$$

Need 2 things: find the intersection ?

then evaluate the integral

Example (cont.)

To get ?, solve

$$\frac{2}{x+1} = x \quad \text{(where the two curves intersect)}$$

$$\frac{2}{\cancel{x+1}} \cdot \cancel{(x+1)} = x(x+1) \quad \text{so} \quad 2 = x^2 + x$$

$$x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0 \quad \text{so} \quad x = -2, 1$$

The intersection was for $x > 0$, so we pick $x = 1$.

Example (cont.)

Now we find $\int_0^1 \left(\frac{2}{x+1} - x\right) dx = \underbrace{\int_0^1 \frac{2}{x+1}\,dx}_{} - \int_0^1 x\,dx$

$\int_0^1 \frac{2}{x+1}\,dx = 2\ln 2$

let $u = x+1$ so $du = dx$

Bounds: $1 \rightsquigarrow 1+1 = 2$

$0 \rightsquigarrow 0+1 = 1$

so $\int_0^1 \frac{2}{x+1}\,dx = \int_1^2 \frac{2}{u}\,du = 2\ln|u| \Big|_1^2$

$= 2[\ln 2 - \cancelto{0}{\ln 1}] = \boxed{2\ln 2}$

$\int_0^1 x\,dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$

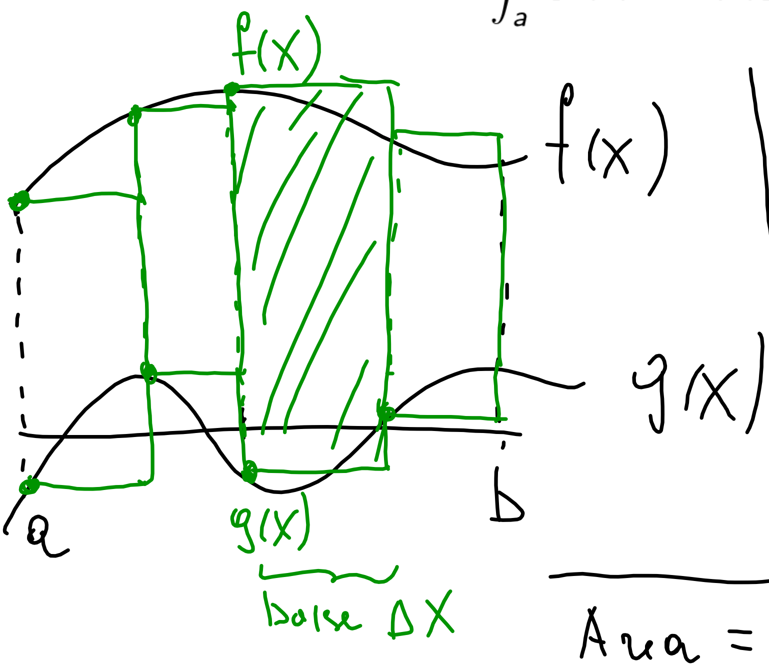
Put this together, area $= 2\ln 2 - \frac{1}{2}$

Area between two curves: general situation Very important!

Theorem. If $f(x)$ and $g(x)$ are continuous with $\boxed{f(x) \geq g(x)}$ on $a \leq x \leq b$, the area between the curves $y = f(x)$ and $y = g(x)$ when $a \leq x \leq b$ is

$$A = \int_a^b [f(x) - g(x)]dx.$$

} same as in previous slide



Area /// = Base · height

$= \Delta x \cdot (f(x) - g(x))$

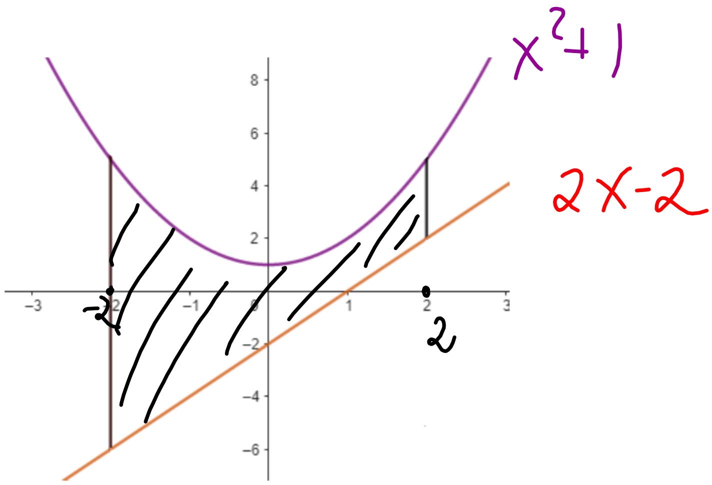
$= (f(x) - g(x))\Delta x$

Take limit as $\Delta x \to 0$,

$$\text{Area} = \int_a^b (f(x) - g(x))\, dx$$

Example

Find the shaded area, if the purple curve is $y = x^2 + 1$ and the line is $y = 2x - 2$.



$$\text{Area} = \int_{-2}^{2} \left[(x^2+1) - (2x-2)\right] dx$$

$$= \int_{-2}^{2} (x^2+1-2x+2)\, dx = \int_{-2}^{2} (x^2-2x+3)\, dx$$

$$= \left(\frac{1}{3}x^3 - x^2 + 3x\right)\Big|_{-2}^{2} = \left(\frac{1}{3}\cdot 2^3 - 2^2 + 3\cdot 2\right)$$

$$- \left(\frac{1}{3}(-2)^3 - (-2)^2 + 3\cdot(-2)\right) = \frac{8}{3} \cancel{-4} + 6 + \frac{8}{3} \cancel{+4} + 6$$

$$= \frac{16}{3} + 12 = \frac{52}{3}$$

Example (cont.)

Example

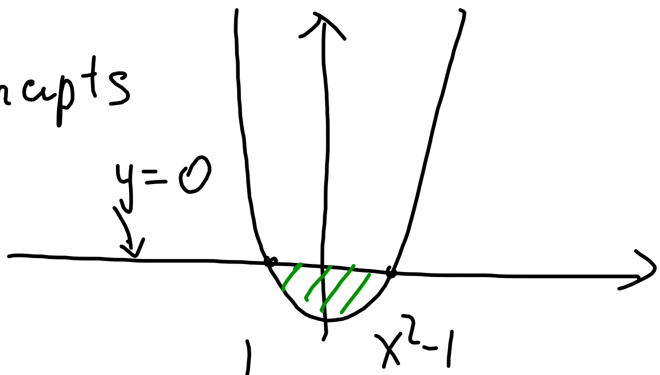
Find the area between $y = x^2 - 1$ and the x-axis.

Soln: Draw the graph, formulate the integral.

Bounds: x-intercepts

$x^2 - 1 = 0$

So $x = -1, 1$



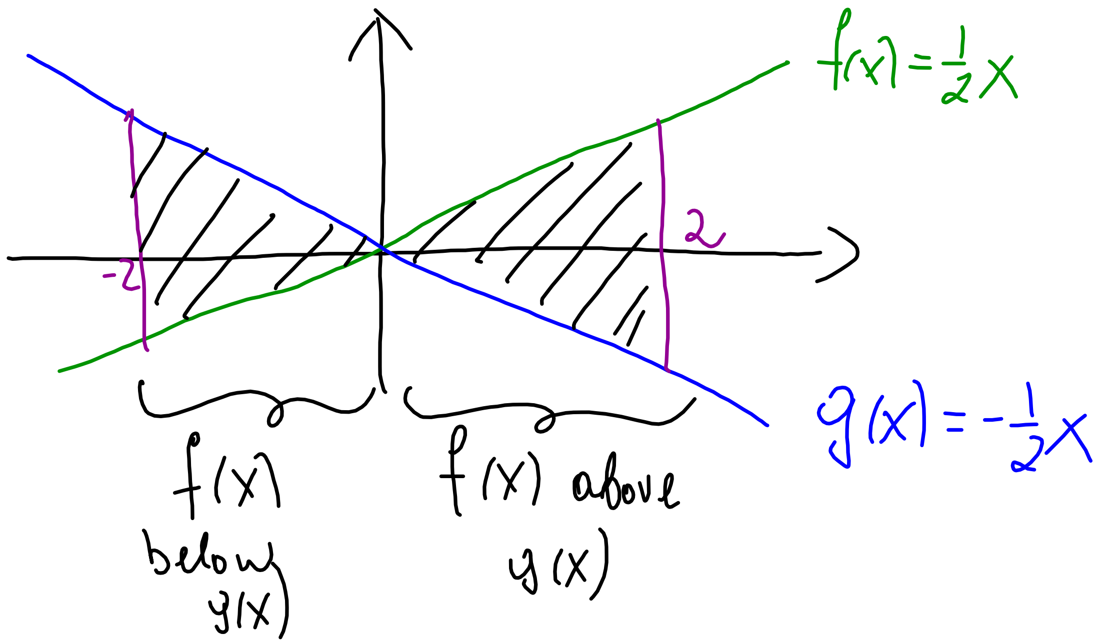
$$\int_{-1}^{1} \left(0 - (x^2 - 1)\right) dx = \int_{-1}^{1} (1 - x^2) dx = \left(x - \frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \left(1 - \frac{1}{3} \cdot 1^3\right) - \left((-1) - \frac{1}{3}(-1)^3\right) = 1 - \frac{1}{3} + 1 - \frac{1}{3}$$

$$= 2 - \frac{2}{3} = \frac{4}{3}$$

Example

Find the area between $f(x) = \frac{1}{2}x$, $g(x) = -\frac{1}{2}x$, $x = -2$ and $x = 2$.



Area is 2 separate pieces:

$$\int_{-2}^{0} (g(x) - f(x))\,dx + \int_{0}^{2} (f(x) - g(x))\,dx$$