McMaster University

FACULTY OF SCIENCE
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5.4 More Applications

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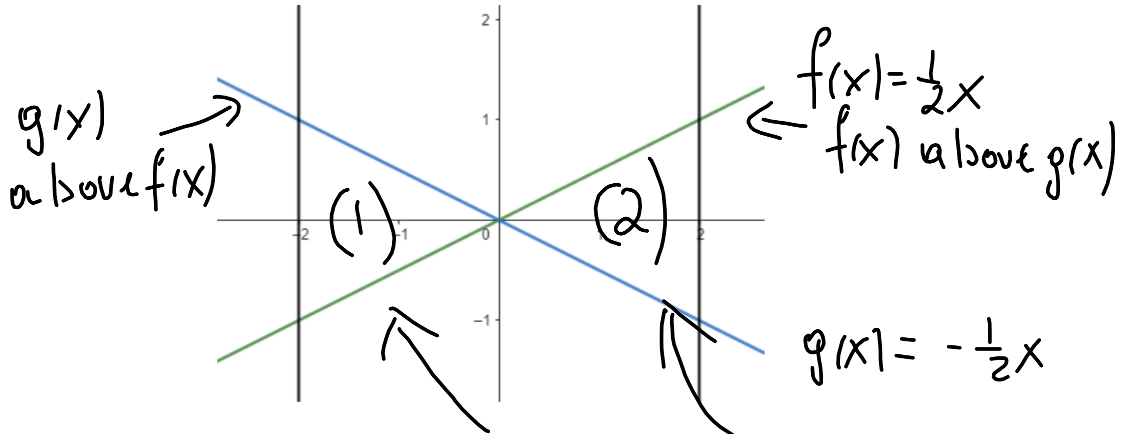
Lecture 24

Finishing example from last time

Find the area between $f(x) = \frac{1}{2}x$, $g(x) = -\frac{1}{2}x$, $x = -2$ and $x = 2$.

Soln.



Area= Area(1)+Area(2) $= \int_{-2}^{0}[g(x)-f(x)]dx + \int_{0}^{2}[f(x)-g(x)]dx$

$$= \int_{-2}^{0}\left(-\frac{1}{2}x - \frac{1}{2}x\right)dx + \int_{0}^{2}\left(\frac{1}{2}x - \left(-\frac{1}{2}x\right)\right)dx$$

$$= \int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx = -\frac{1}{2}x^2\Big|_{-2}^{0} + \frac{1}{2}x^2\Big|_{0}^{2}$$

$$= -\frac{1}{2}\left(0^2 - (-2)^2\right) + \frac{1}{2}\left(2^2 - 0^2\right) = \frac{4}{2} + \frac{4}{2} = 4$$

Example (cont.)

Alternatively, the 2 regions have the same area (same inverted triangle),

$$\text{Area} = 2 \cdot \text{Area}(2) = 2\int_0^2 \left(\tfrac{1}{2}x - \left(-\tfrac{1}{2}x\right)\right)dx$$

$$= 2 \cdot \int_0^2 x\,dx = \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot x^2 \Big|_0^2 = 2^2 - 0^2$$

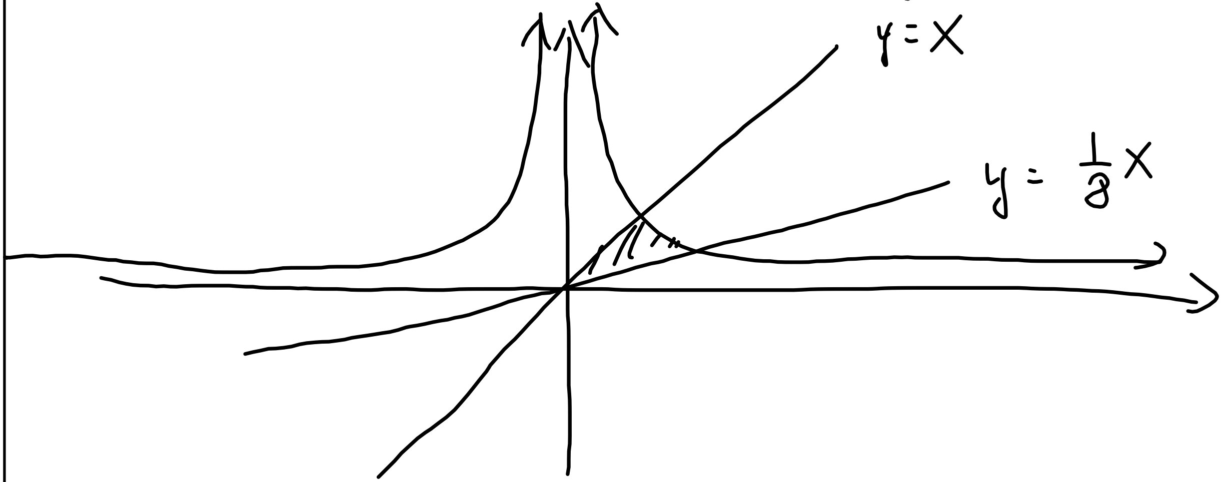
$$= 4$$

Another example

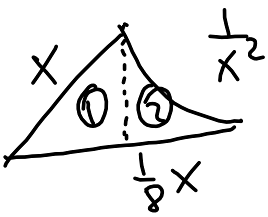
Find the area between the curves

$$y = \frac{1}{x^2}, \qquad y = x, \qquad \text{and} \qquad y = \frac{x}{8} \cdot = \frac{1}{8}x$$

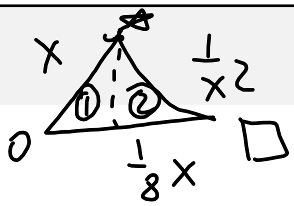
Soln: Sketch graphs, find region.



Zoom in:



Example (cont.)



Area ①: $y = x$ is above $y = \frac{1}{8}x$

so $\displaystyle\int_0^{\star} \left[x - \frac{1}{8}x \right] dx$

Find $\star$: intersection of x and $\frac{1}{x^2}$

Solve for x: $x = \frac{1}{x^2}$, so $x^3 = 1$ so $x = 1$

Area ① $= \displaystyle\int_0^1 \left(x - \frac{1}{8}x\right) dx = \int_0^1 \frac{7}{8}x\, dx = \frac{7}{8} \cdot \frac{1}{2} x^2 \Big|_0^1$

$= \frac{7}{16}\left(1^2 - 0\right) = \frac{7}{16}$

Area ②: $y = \frac{1}{x^2}$ is above $\frac{1}{8}x$

Example (cont.)

$$\text{Area } ② = \int_{\star=1}^{\square} \left(\frac{1}{x^2} - \frac{1}{8}x\right) dx$$

Find $\square$: intersection b/w $y = \frac{1}{x^2}$ and $\frac{1}{8}x$

$\frac{1}{x^2} = \frac{1}{8}x$ so $1 = \frac{1}{8}x^3$ so $x^3 = 8$

so $x = 2$

$$\text{Area } ② = \int_1^2 \left(\frac{1}{x^2} - \frac{1}{8}x\right) dx = \int_1^2 \left(x^{-2} - \frac{1}{8}x\right) dx$$

$$= \left(\frac{1}{-2+1} x^{-2+1} - \frac{1}{8} \cdot \frac{1}{2} x^2\right)\Bigg|_1^2 = \left(-\frac{1}{x} - \frac{1}{16}x^2\right)\Bigg|_1^2$$

$$= \left(-\frac{1}{2} - \frac{1}{16} \cdot 2^2\right) - \left(-1 - \frac{1}{16}\right) = \frac{5}{16}$$

Total area is $\frac{7}{16} + \frac{5}{16} = \frac{12}{16} = \frac{3}{4}$

Additional practice problems for finding areas between curves:
Ex. 1, 3, 5, 7, 9, 11, 15, 17, 49 in Section 5.4.

Averages

Suppose we have a sequence $1, 2, 3, \ldots, n$. How do we find the average of this sequence?

Sum them up, divide by # items.

$$\overbrace{1 + \underbrace{2 + 3 + 4 + \ldots (n-1)}_{n+1} + n}^{n+1} = \frac{(n+1)\cdot n}{2}$$

Avg is $\frac{(n+1)\cdot \cancel{n}}{2} \cdot \frac{1}{\cancel{n}} = \frac{n+1}{2}$

Section 5.4: Application 2: The average value of a function

Now supose we have the continuous counterpart $f(x) = x$, defined not only for integer numbers, but for all real numbers. How do we find the average value of $f(x)$ on $1 \leq x \leq n$?

x	1 ~ 2	- -	n
$f(x)$	1 ~ 2	- -	n

Same idea: we sum "infinitely" many values $\underbrace{\text{limit}}$ becomes an integral

Divide by length of interval

Here: $\frac{1}{n-1} \cdot \int_1^n x\,dx = \frac{1}{n-1} \cdot \frac{1}{2} x^2 \Big|_1^n$

$= \frac{1}{n-1} \cdot \frac{1}{2}(n^2-1) = \frac{1}{\cancel{n-1}} \cdot \frac{1}{2}(n+1)\cancel{(n-1)}$

$= \frac{n+1}{2}$

The average value of a function (cont.)

Theorem. If $f(x)$ is continuous on $a \leq x \leq b$, then its average value is

$$\textbf{Avg} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Example. Find the average value of $f(x) = 5x^2 + 2x + 5$ on $1 \leq x \leq 4$.

Soln: $\frac{1}{4-1}\int_1^4 (5x^2+2x+5)\,dx$

$= \frac{1}{3}\left(\frac{5}{3}x^3 + x^2 + 5x\right)\Big|_1^4$

$= \frac{1}{3}\left[\left(\frac{5}{3}\cdot 4^3 + 4^2 + 5\cdot 4\right) - \left(\frac{5}{3}\cdot 1^3 + 1^2 + 5\cdot 1\right)\right]$

$= \frac{1}{3}\left[\frac{428}{3} - \frac{23}{3}\right] = 45$