McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

6.1: Integration by parts

Angelica Babei

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Lecture 25

Chapter 6.1: Integration by parts

Chain rule $\longleftrightarrow$ u-substitution
Product rule $\longleftrightarrow$ integration by parts

Recall the product rule:

$$[u(x)v(x)]' = u(x)v'(x) + v(x)u'(x)$$

Integrate both sides with respect to x:

$$\int [u(x)v(x)]'dx = \boxed{\int u(x)v'(x)dx} + \int v(x)u'(x)dx$$
$$u(x)v(x) = \int u(x)v'(x)dx + \int v(x)u'(x)dx$$

Solving for $\int u(x)v'(x)dx$ gives

$$\boxed{\int u(x)v'(x)dx} = u(x)v(x) - \boxed{\int v(x)u'(x)dx}$$

The idea is that we want ↑ to be easier than $\boxed{}$

Integration by parts formula(s)

$$\int u(x)\boxed{v'(x)dx} = u(x)v(x) - \int v(x)\boxed{u'(x)dx} \qquad \bigstar$$

A friendlier form to remember:

$$v'(x) = \frac{dv}{dx} \quad \rightarrow \quad \underline{dv} = \boxed{v'(x)dx}$$

$$u'(x) = \frac{du}{dx} \quad \rightarrow \quad \underline{du} = \boxed{u'(x)dx}$$

Rewrite the earlier formula $\bigstar$ as

$$\int u\underline{dv} = uv - \int v\underline{du}$$

In definite integrals,

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

Example

$\boxed{\int u\,dv = uv - \underbrace{\int v\,du}}$ needs to be easy to solve

$x \cdot e^x$

$$\int xe^x\,dx =$$

2 options: ① $u(x) = x$, $v'(x) = e^x$

② $u(x) = e^x$, $v'(x) = x$

Try ①: If $u(x) = x$, $u'(x) = 1$ so $du = dx$

$v'(x) = e^x$ so $v(x) = \int v'(x)\,dx = \int e^x\,dx = e^x$

$\boxed{\int v\,du = \int \underbrace{e^x}_{v} \cdot \underbrace{dx}_{du}}$, simpler than $\int xe^x\,dx$

Try ②: If $u(x) = e^x$, $u'(x) = e^x = \frac{du}{dx}$

so $du = e^x dx$

$v'(x) = x$, then

$v(x) = \int x\,dx = \frac{1}{2}x^2$

Then $\boxed{\int v\,du = \int \frac{1}{2}x^2 \cdot e^x\,dx}$, worse than $\int xe^x\,dx$:(

So we choose ① !

Example (cont.)

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Chose ①: $u(x) = x$, $du = dx$

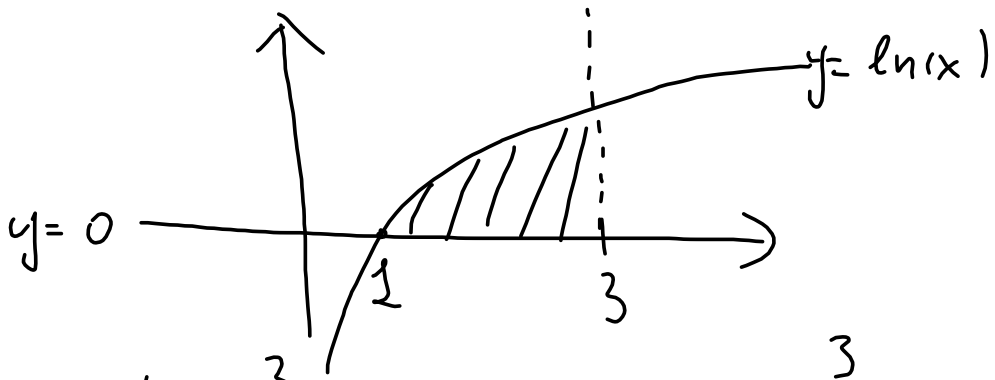
$v'(x) = e^x$, $v(x) = e^x$

Then $\int \underbrace{x}_{u}\,\underbrace{e^x dx}_{dv} = \underbrace{x}_{u}\cdot\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\cdot\underbrace{dx}_{du}$

$= xe^x - e^x + C = \underbrace{e^x(x-1) + C}_{\text{Answer.}}$

Example

Find the area between $y = \ln x$, the x-axis, and $x = 3$.



Need $\int_1^3 (\ln(x) - 0)\,dx = \int_1^3 1\cdot\ln(x)\,dx$

2 options: ① $u(x) = 1$, $v'(x) = \ln(x)$
② $u(x) = \ln(x)$, $v'(x) = 1$

First case: ① $v(x) = \int v'(x)\,dx = \int \ln(x)\,dx$, this is what we want to solve!!

② $u(x) = \ln(x)$ so $du = \frac{1}{x}\,dx$
$v'(x) = 1$ so $v(x) = \int 1\,dx = x$
Then $\int v\,du = \int \underbrace{\cancel{x}}_{v} \cdot \underbrace{\frac{1}{\cancel{x}}\,dx}_{du} = \int 1\,dx = x + C$

Example (cont.)

Chose (2): $u(x) = \ln(x)$, $du = \frac{1}{x}dx$

$\frac{dv}{dx} = v'(x) = 1$, $v(x) = x$

$$\int_1^3 \underbrace{\ln(x)}_{u} \cdot \underbrace{1 \cdot dx}_{dv} = \underbrace{\ln(x)}_{u} \cdot \underbrace{x}_{v} \Big|_1^3 - \int_1^3 \underbrace{x}_{v} \cdot \underbrace{\frac{1}{x}dx}_{du}$$

$$= x\ln(x)\Big|_1^3 - x\Big|_1^3$$

$$= 3\ln(3) - 1 \cdot \underbrace{\ln(1)}_{0} - (3-1) = \boxed{3\ln 3 - 2}$$

Example

What is the average value of $f(x) = 2x^2 e^x$ on $0 \leq x \leq 5$?

Soln: $\frac{1}{5-0} \int_0^5 2x^2 e^x \, dx = \frac{2}{5} \int_0^5 x^2 e^x \, dx$

Choose: $u(x) = x^2$, $v'(x) = e^x$

$du = 2x\,dx$, $v(x) = \int e^x dx = e^x$

$$\boxed{\frac{2}{5}} \int_0^5 \underbrace{x^2}_{u} \underbrace{e^x dx}_{dv} = \frac{2}{5} \left(x^2 \cdot e^x \Big|_0^5 - \int_0^5 \underbrace{e^x}_{v} \underbrace{2x\,dx}_{du} \right)$$

$$= \frac{2}{5} \left(x^2 e^x \Big|_0^5 - 2 \underbrace{\int_0^5 x e^x \, dx}_{\text{Did in ex } 1} \right)$$

Example (cont.)

$$= \frac{2}{5}\left(x^2 e^x \Big|_0^5 - 2 \cdot \overbrace{e^x (x-1) \Big|_0^5}^{ex\ 1} \right)$$

$$= \frac{2}{5}\left(5^2 e^5 - \cancel{0^2 e^0} - \left[2e^5(5-1) - 2 \cdot e^0 \cdot (0-1) \right] \right)$$

$$= \frac{2}{5}\left(25e^5 - 8e^5 - 2\right) \approx 1008.4$$

Example (cont.)

Some observations

$\int u\,dv = uv - \int v\,du$

- Always need to be able to get $\int v'(x)\,dx = v(x)$
  (e.g. ex 2)
- $\int v\,du$ need to be easier than $\int u\,dv$
  (e.g: $u(x)$ a polynomial, then du has smaller degree, simpler)
- In ex 3, you might have to use method multiple times.

Take a minute to digest!

Take a minute to jot down what you can remember from these examples. For example, what would be some specific good candidates for $u(x)$ and $v'(x)$?

Additional practice problems for integration by parts:
Ex. 1, 3, 5, 7, 11, 17, 19, 21, 23, 25, 39, 43, 45, 55, 57, 59 in Section 6.1