FACULTY OF SCIENCE
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5.5: Continuous income streams

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Lecture 27

u-sub or by parts?

1. $\int \boxed{(6x+5)}(3x^2+5x-1)^{3/2}dx$: u-sub.

$u = 3x^2+5x-1 \rightsquigarrow \frac{du}{dx} = \boxed{6x+5}$

2. $\int \underbrace{(3x-7)}_{???}\sqrt[7]{6-\frac{1}{6}x}\,dx$: either method

u-sub: $u = 6-\frac{1}{6}x$ so $du = -\frac{1}{6}dx$

$6u = 36 - x$

so $x = 36-6u$

by parts: $u(x) = 3x-7 \rightsquigarrow du = 3dx$

$v'(x) = \sqrt[7]{6-\frac{1}{6}x}$ so $v(x) = \int \underbrace{(6-\frac{1}{6}x)^{1/7}}_{\text{use u-sub.}}dx$

Defn: (from 5.3) If $Q'(x)$ is continuous on $a \le x \le b$, then the net change in $Q(x)$ as x varies from $x=a$ to $x=b$ is given by

$$Q(b)-Q(a) = \int_a^b Q'(x)\,dx$$

Section 5.5: Continuous income streams

Recall the formula for the balance of continuous compounding:

$$B(t) = \underbrace{P_0}_{\text{principal}} e^{rt} \quad r \to \text{interest rate}$$

In this basic case, money is deposited once, at the beginning at time $t = 0$. It is more common to have recurring deposits, such as every month, week, day, etc.

On the extreme, we can have money flowing continuously (in what scenarios would this model make sense?). How can we calculate the present or future value of the investment?

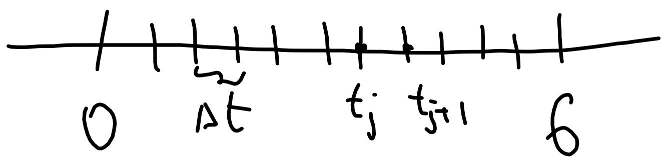
present: how much is it worth today

future: the amount of money it generates at some point in the future.

Motivating example

Money is transferred continuously to an account at the constant rate of \$2,400 per year. The account earns interest at the rate of 6% compounded continuously. How much money will this account have in 6 years?

Expanded soln: We approximate first



Divide the interval into small pieces

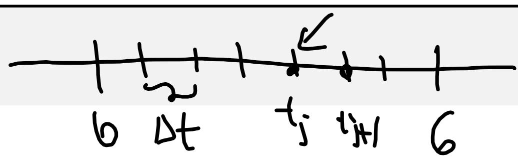
Δt = length of subdivision

Qn: How much money was deposited in the Δt time period?

$$M = 2400 \,(\text{dollars}/\cancel{\text{year}}) \cdot \Delta t \,(\cancel{\text{years}})$$

$$= 2400\,\Delta t \,(\text{dollars})$$

Example (cont.)



Suppose we deposit $M = 2400\,\Delta t$ at the beginning of the interval $[t_j, t_{j+1}]$

Qn: For how long will this deposit gain interest?

Answer: For $(6 - t_j)$ (years)

Qn: What is the future value of this deposit at the end of the 6 years?

Answer: Formula for balance is $(P_0 \cdot e^{rt})$

Here, it is

$$\underbrace{2400\,\Delta t}_{\text{deposit } M} \cdot e^{0.06(6-t_j)}$$

Example (cont.)

$2400 \cdot e^{0.06(6-t_j)} \Delta t$

To get the future value of the whole investment (all the subdivisions) we sum them up AND take limit as $\Delta t \to 0$

Sum over subdivisions $\rightsquigarrow \int_0^6$

$t_j \rightsquigarrow t$ $\qquad \Delta t \rightsquigarrow dt$

Sum over $2400 e^{0.06(6-t_j)} \Delta t \rightsquigarrow \int_0^6 2400 e^{0.06(6-t)} dt$

$$\int_0^6 2400 e^{0.06(6-t)} dt = 2400 \int_0^6 e^{.06 \cdot 6} \cdot \underline{e^{-0.06t}} dt$$

$$= 2400 \cdot e^{0.36} \frac{1}{-0.06} e^{-0.06t} \Big|_0^6$$

$$= -40000 \, e^{0.36} \cdot (e^{-0.36} - e^0)$$

$$\approx 17333.18$$

Formula for future value of continuous income stream

For a continuous income stream $f(t)$ over a time period $0 \leq t \leq T$ and an annual interest rate r compounded continuously, the future value of the income stream over the term T (i.e. the balance after T years) is

$$\boxed{FV = \int_0^T f(t)e^{r(T-t)}dt = e^{rT}\int_0^T f(t)e^{-rt}dt.}$$

The present value of continuous income stream

The present value is the amount to be deposided now at the prevailing interest rate to give the same future value as the income stream.

Recall that given a principal P, interest rate r, the balance after T years is $B(T) = Pe^{rT}$.

Earlier today, we saw that given a continuous income stream $f(t)$, its future value is

$$B(T) = e^{rT} \int_0^T f(t)e^{-rt}\,dt.$$

To find the present value, set these two equal, solve for P (the present value):

$$P\cancel{e^{rT}} = \cancel{e^{rT}} \int_0^T f(t)e^{-rt}\,dt.$$

Formula for present value of continuous income stream

For a continuous income stream $f(t)$ over a time period $0 \leq t \leq T$ and an annual interest rate r compounded continuously, the present value of the income stream over the term T is

$$\boxed{PV = \int_0^T f(t)e^{-rt}\,dt.}$$

So $FV = PV \cdot e^{rT}$ (just as in chapter 4)

Example

Money is transferred to an account at the continuous rate of $R(t) = 1000te^{-0.3t}$ dollars per year for 5 years. The account pays 4% interest, compounded continuously.

(a) How much is this investment of 5 years worth now? $= PV$

$$PV = \int_0^5 \underbrace{1000te^{-0.3t}}_{\text{deposits } R(t)} \cdot \underbrace{e^{-0.04t}}_{\text{interest rate}} dt$$

$$= 1000 \int_0^5 t e^{-0.34t} dt.$$ Solve by parts

$u(t) = t$

so $du = dt$

$v'(t) = e^{-0.34t}$

so $v(t) = \int e^{-0.34t} dt$

$= \frac{1}{-0.34} e^{-0.34t}$

Example (cont.)

$$= 1000\left(t \cdot \frac{1}{-0.34} e^{-0.34t} \Big|_0^5 - \int_0^5 \frac{1}{-0.34} e^{-0.34t}\, dt \right)$$

$$= 1000\left(5 \cdot \left(-\frac{1}{0.34}\right) e^{-1.7} - 0 - \frac{1}{(0.34)^2} e^{-0.34t} \Big|_0^5 \right)$$

$$= 1000\left(-\frac{5e^{-1.7}}{0.34} - \frac{1}{(0.34)^2} e^{-1.7} + \frac{1}{(0.34)^2} e^{0} \right)$$

$$\approx 4383.69$$

Example (cont.)

(b) In the same example as before, how much money will the account accumulate over the 5 years?

$$FV = e^{0.04 \cdot 5} \underbrace{\int_0^5 1000\, t\, e^{-0.3t} \cdot e^{-0.04t}\, dt}_{\text{done in } (a)}$$

$$\approx e^{0.2} \cdot 4383.69 \approx 5354.25$$