McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

7.1: Functions of multiple variables

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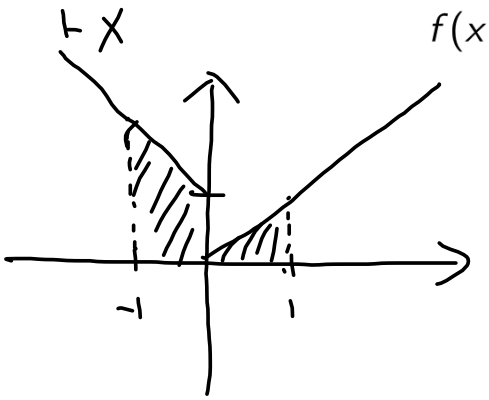
Lecture 28

Loose thread from definite integrals: piecewise functions

Example. Find $\int_{-1}^{1} f(x)dx$, where

$$f(x) = \begin{cases} 1-x & x < 0 \\ x & x \geq 0 \end{cases}$$



① $f(x)$ above the x-axis

Can interpret $\int_{-1}^{1} f(x)dx$ as the area of the shaded region.

From the pic, area is $1.5 + 0.5 = 2$.

Or, as a sum of two integrals:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} (1-x)dx + \int_{0}^{1} x\,dx$$

$$= \left(x - \frac{1}{2}x^2\right)\Big|_{-1}^{0} + \frac{1}{2}x^2\Big|_{0}^{1}$$

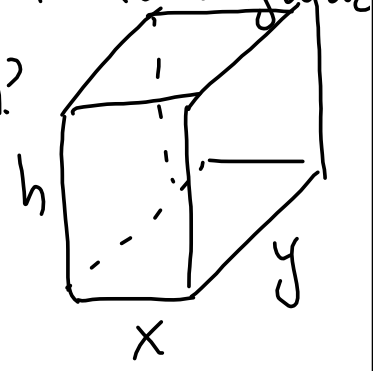
$$= \left(0^2 - \frac{1}{2}\cdot 0^2\right) - \left((-1) - \frac{1}{2}\cdot(-1)^2\right) + \left(\frac{1}{2}\cdot 1^2 - \frac{1}{2}\cdot 0^2\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{2} = 2.$$

Chapter 7: Functions of multiple variables

So far: We have worked with functions with only one independent variable, e.g. $f(x) = x^2 + 5x - 2e^x$.
Today: Functions $f(x, y)$ or $g(x, y, z)$ etc. with multiple independent variables, e.g. $f(x, y) = x^2 + 5xy - 3y^2$.

Ex: What is the volume of a rectangular box with width x, length y, height h?



$V(x, y, h) = xyh$

height, width and length don't depend on each other.

Any $x, y, h > 0$ make sense.

Another simple example

The output at a certain factory is

$$Q(K, L) = 120K^{2/3}L^{1/3} \quad \text{(units)},$$

where K is the capital investment, measured in units of \$1000, and L is the size of the labor force, measured in worker hours. What is the output if the capital investment is \$125000, and the size of the labor force is 1331 hours?

$\frac{125000}{1000} = K$

L

Answer: We plug in $K = 125$, $L = 1331$

$$Q(125, 1331) = 120 \cdot (125)^{2/3} \cdot (1331)^{1/3}$$
$$= 120 \cdot (5^3)^{2/3} \cdot (11^3)^{1/3}$$
$$= 120 \cdot 5^2 \cdot 11 = 33000 \text{ (units)}$$

Any $K, L \geq 0$ make sense.

Definition

A function f of the two variables x and y is a rule that assigns to each ordered pair (x, y) in a given set D, the domain of f, exactly one real value, denoted by $f(x, y)$.

This definition immediately generalizes to functions of 3 or more variables.

e.g. $g(x, y, z)$

$h(a, b, c, d)$

The domain of $f(x, y)$

The domain of $f(x, y)$ is the set of pairs of values (x, y) for which $f(x, y)$ makes sense. It can be convenient to think of the domain as a region in the xy-plane.

Ex: Domain of $f(x,y) = \sqrt{9 - x^2 - y^2}$

Cannot take $\sqrt{\ }$ of negative numbers

So $9 - x^2 - y^2 \geq 0$