McMaster University
FACULTY OF SCIENCE
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# 7.2: Partial derivatives

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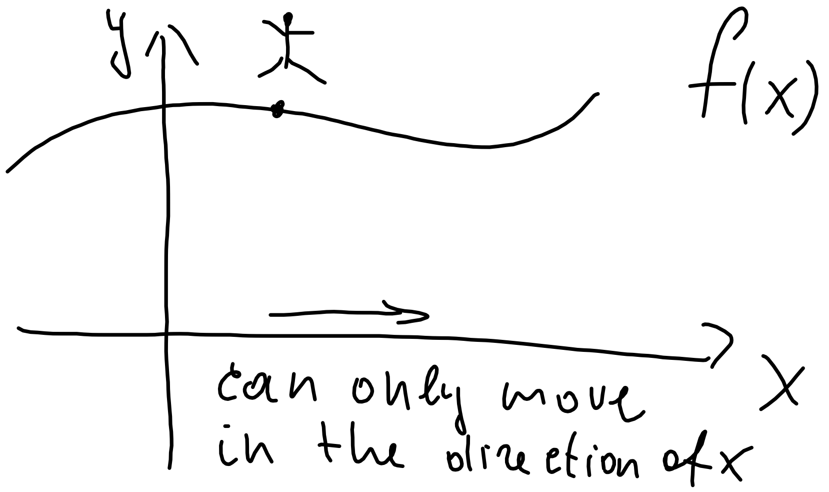
## 7.2: Partial derivatives

How does differentiation work for 2 or more independent variables?
Recall that the derivative = rate of change.
In functions of one variable, as we increase $x$, the change in $y$ is

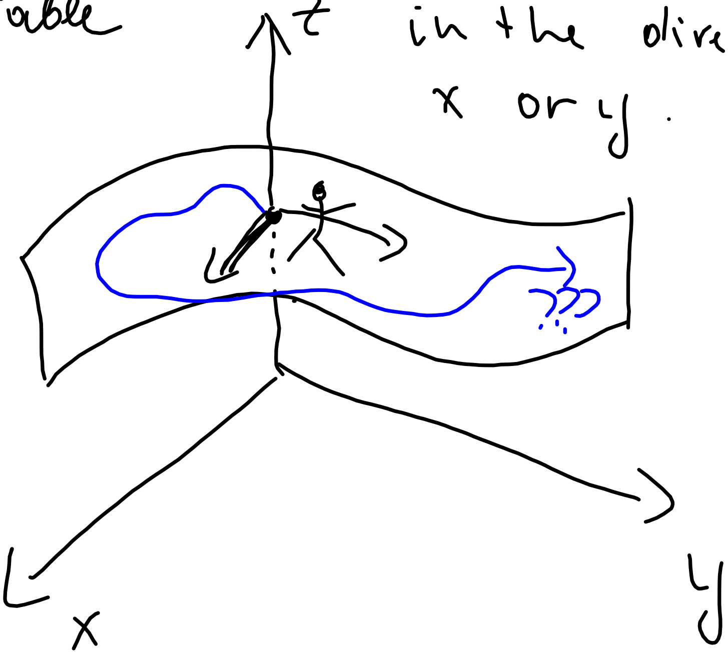
$$\frac{dy}{dx} = \frac{df}{dx} = f'(x)$$



## Rates of change for two independent variables

With 2 or more variables, the rate of change depends on which $\underbrace{\text{direction you are moving}}_{\text{which variable changes}}$.

We can move in the direction of x or y.




Remark: If we move in the x direction, y stays constant (fixed), similarly, if we move in the y-direction, x stays constant.

## Example

The demand for a product is

$$D(x, y) = 200 - 10x^2 + 20xy \text{ (units per month)},$$

where $x$ is the price of the product and $y$ is the price of the competing product.

(1) What is the rate of change of demand with respect to the price of the product? (w.r.t. $x$)

Idea: Find the derivative w.r.t. $x$, treat $y$ as a constant.

$$\underbrace{D_x}_{\text{direction}}(x,y) = 0 - 10\cdot(2x) + 20\cdot 1\cdot y$$

$$= -20x + 20y$$

## Example (cont.)

$$D(x, y) = 200 - 10x^2 + 20xy \text{ (units per month)},$$

where $x$ is the price of the product and $y$ is the price of the competing product.

(2) What is the rate of change of $D(x, y)$ with respect to the price of the competing product? ( w.r.t. $y$)

Now we differentiate w.r.t. $y$, while treating $x$ as a constant.

treat $10x^2$ as a constant

$$\underbrace{D_y}_{\text{direction}}(x, y) = 0 - 0 + 20 \cdot x \cdot 1$$

$$= 20x$$

## Partial derivatives

Suppose $z = f(x, y)$. The partial derivative of $f$ with respect to $x$ is denoted by one of the following

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x},$$

$\frac{\partial f}{\partial x}$

and is the function of $x$ and $y$ obtained by differentiating $f$ with respect to $x$, while treating $y$ as a constant.
The partial derivative with respect to $y$ is denoted by

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y},$$

and is obtained by differentiating $f$ with respect to $y$ while treating $x$ as a constant.
These are called the "first-order" partial derivatives.

## Example I

Find the first-order partial derivatives of

$$f(x, y) = 5x^2y + 2xy^3 + 3y^2$$

$$f_x(x,y) \underset{\substack{\text{treat} \\ y \text{ as} \\ \text{constant}}}{=} \boxed{\frac{\partial}{\partial x}}(5x^2y + 2xy^3 + 3y^2)$$

$$= 5 \cdot (2x) \cdot y + 2 \cdot 1 \cdot y^3$$

$$+ 0 = 10xy + 2y^3$$

$$f_y(x,y) \underset{\substack{\text{treat} \\ x \text{ as} \\ \text{constant}}}{=} \frac{\partial}{\partial y}(5x^2y + 2xy^3 + 3y^2)$$

$$= 5x^2 \cdot 1 + 2x(3y^2) + 3 \cdot (2y)$$

$$= 5x^2 + 6xy^2 + 6y$$

## Example II

Find the first-order partial derivatives of

$$f(s,t) = \frac{3t}{2s} = \frac{3}{2} t s^{-1}$$

$$f_t(s,t) = \frac{\partial}{\partial t}\left(\frac{3}{2} t s^{-1}\right) = \frac{3}{2} \cdot 1 \cdot s^{-1}$$

$$= \frac{3}{2s}$$

$$f_s(s,t) = \frac{\partial}{\partial s}\left(\frac{3}{2} t s^{-1}\right)$$

$$= \frac{3}{2} t \cdot (-1) s^{-2} = -\frac{3t}{2s^2}$$

$$= -\frac{3}{2} t s^{-2}$$

## Higher-order partial derivatives

These are analogous to higher-order derivatives, and we differentiate partial derivatives further. We write the partial derivative of $f_x$ with respect to $x$ as

$$f_{xx} = (f_x)_x = \boxed{\frac{\partial^2 f}{\partial x^2}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

**Example:**

$$f(x,y) = 5x^2y + 2xy^3 + 3y^2$$
$$\longrightarrow \quad f_x(x,y) = 10xy + 2y^3$$
$$f_y(x,y) = 5x^2 + 6xy^2 + 6y$$

$$f_{xx} = \frac{\partial}{\partial x}(10xy + 2y^3) = 10 \cdot 1 \cdot y + 0$$
$$= 10y$$

## Higher-order partial derivatives (cont.)

The partial derivative of $f_x$ with respect to $y$ is denoted by

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

**Example:**

$$f(x,y) = 5x^2y + 2xy^3 + 3y^2$$
$$f_x(x,y) = 10xy + 2y^3$$
$$f_y(x,y) = 5x^2 + 6xy^2 + 6y$$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(10xy + 2y^3\right)$$
$$= 10x \cdot 1 + 2 \cdot (3y^2) = 10x + 6y^2$$

## Higher-order partial derivatives (cont.)

Similarly, we can get the partial derivatives of $f_y$ with respect to $x$ and with respect to $y$. In total, we have 4 **second-order** partial derivatives:

$$\begin{tikzcd}
 & & f_{xx} \\
 & f_x \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & \\
f(x,y) \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & f_{xy} \\
 & f_y \arrow[r, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & f_{yx} \\
 & & f_{yy}
\end{tikzcd}$$

They are equal! $f_{xy} = f_{yx}$

Earlier example: $f_y(x,y) = 5x^2 + 6xy^2 + 6y$

$$f_{yx} = \frac{\partial}{\partial x}(5x^2 + 6xy^2 + 6y) = 10x + 6y^2 + 0$$
$$= f_{xy}$$

$$f_{yy} = \frac{\partial}{\partial y}(5x^2 + 6xy^2 + 6y) = 0 + 12xy + 6$$