



FACULTY OF SCIENCE
Department Of Mathematics & Statistics

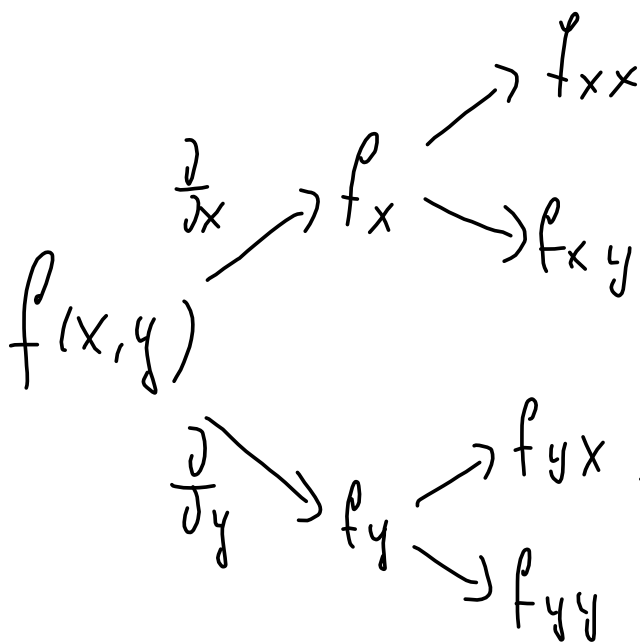
7.2: The chain rule

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Lecture 31



Higher-order partial derivatives



These (e.g. f_{xy}, f_{yx}) are called "mixed 2nd order partials". In general (this course) they are equal $f_{xy} = f_{yx}$. It doesn't matter if you first differentiate w.r.t y , then w.r.t. x , or first w.r.t x then y .

This philosophy holds for higher order partials, e.g. $f_{xyy} = f_{yxy} = f_{yyx}$

But, $f_{xyy} \neq f_{yxx}$

Another example

$$f(x, y) = (y + 2) \ln(x^2 + y^3)$$

Evaluate the second order partial derivatives at $(1, 0)$. $(x=1, y=0)$

$$\begin{aligned} f_x &= \frac{\partial}{\partial x} \left((y+2) \ln(x^2 + y^3) \right) = (y+2) \frac{\partial}{\partial x} (\ln(x^2 + y^3)) \\ &= (y+2) \cdot \frac{\frac{\partial}{\partial x} (x^2 + y^3)}{x^2 + y^3} = \frac{(y+2) \cdot 2x}{x^2 + y^3} \end{aligned}$$

$$f_y = \frac{\partial}{\partial y} \left((y+2) \cdot \ln(x^2 + y^3) \right) = \text{(use the product rule)}$$

$$\begin{aligned} &= (y+2) \frac{\partial}{\partial y} (\ln(x^2 + y^3)) + \ln(x^2 + y^3) \cdot \frac{\partial}{\partial y} (y+2) \\ &= (y+2) \cdot \frac{\frac{\partial}{\partial y} (x^2 + y^3)}{x^2 + y^3} + \ln(x^2 + y^3) \cdot 1 \\ &= \frac{(y+2) \cdot 3y^2}{x^2 + y^3} + \ln(x^2 + y^3) \end{aligned}$$

Example (cont.) $f_x = \frac{(y+2) \cdot 2x}{x^2+y^3}$ \rightarrow Both have x
 \rightarrow (and y), so use
 quotient rule for
 f_{xx} and f_{xy} .

$$f_{xx} = \frac{LO \cdot dHI - HI \cdot dLO}{(X^2+Y^3)^2}$$

$$= \frac{(x^2+y^3) \cdot \frac{\partial}{\partial x}((y+2) \cdot 2x) - (y+2) \cdot 2x \cdot \frac{\partial}{\partial x}(x^2+y^3)}{(x^2+y^3)^2}$$

$$= \frac{(x^2+y^3) \cdot (y+2) \cdot 2 - (y+2) \cdot 2x \cdot 2x}{(x^2+y^3)^2}$$

$$f_{xy} = (f_{yx}) = \frac{(x^2+y^3) \cdot \frac{\partial}{\partial y}((y+2) \cdot 2x) - (y+2) \cdot 2x \cdot \frac{\partial}{\partial y}(x^2+y^3)}{(x^2+y^3)^2}$$

$$= \frac{(x^2+y^3) \cdot (2x) - (y+2) \cdot 2x \cdot 3y^2}{(x^2+y^3)^2}$$

$$f_{yy} = \frac{\partial}{\partial y} \left(\frac{(y+2) \cdot 3y^2}{x^2+y^3} + \ln(x^2+y^3) \right)$$

$$= \frac{(x^2+y^3) \cdot \frac{\partial}{\partial y}((y+2) \cdot 3y^2) - (y+2) \cdot 3y^2 \cdot \frac{\partial}{\partial y}(x^2+y^3)}{(x^2+y^3)^2} + \frac{\frac{\partial}{\partial y}(x^2+y^3)}{x^2+y^3}$$

$$= \frac{(x^2+y^3) \cdot (9y^2 + 12y) - (y+2) \cdot 3y^2 \cdot 3y^2}{(x^2+y^3)^2} + \frac{3y^2}{x^2+y^3}$$

$$\text{At } (1,0), f_{xx}(1,0) = \frac{2(1^2+0^3) \cdot (0+2) - 2 \cdot 2 \cdot 1 \cdot (0^2)}{(1^2+0^3)^2}$$

$$= -4$$

$$f_{xy}(1,0) = \frac{2 \cdot 1 \cdot (1^2+0^3) \cdot 1 - 2 \cdot 1 \cdot (0+2) \cdot 3 \cdot 0^2}{(1^2+0^3)^2}$$

$$= 2$$

$$f_{yy}(1,0) = 0$$

Additional practice problems for basics of functions of 2 variables:
Exercises 1,3,5,7,9, 17, 19, 21, 23, 25, 27, 29, 31, 35 in Section 7.1
Additional practice problems for finding partial derivatives:
Exercises 1, 3, 7, 11, 13, 15, 17, 21, 23, 25, 27, 29, 31, 33 in
Section 7.2



When the two variables x and y depend on a third one

Recall the demand function

$$D(x, y) = 200 - 10x^2 + 20xy$$

x = price of the product, y = price of the competing product.
Suppose that it is estimated that t months from now, the prices will be

$$x(t) = 10 + 0.5t, \quad y(t) = 12.8 + 0.2t^2.$$

At what rate will the demand change with respect to time 4 months from now?

2 ways to answer the question:

1. We can rewrite $D(x, y)$ as a function of t , by substituting $x(t) = 10 + 0.5t$ and $y(t) = 12.8 + 0.2t^2$ in the expression for $D(x, y)$:

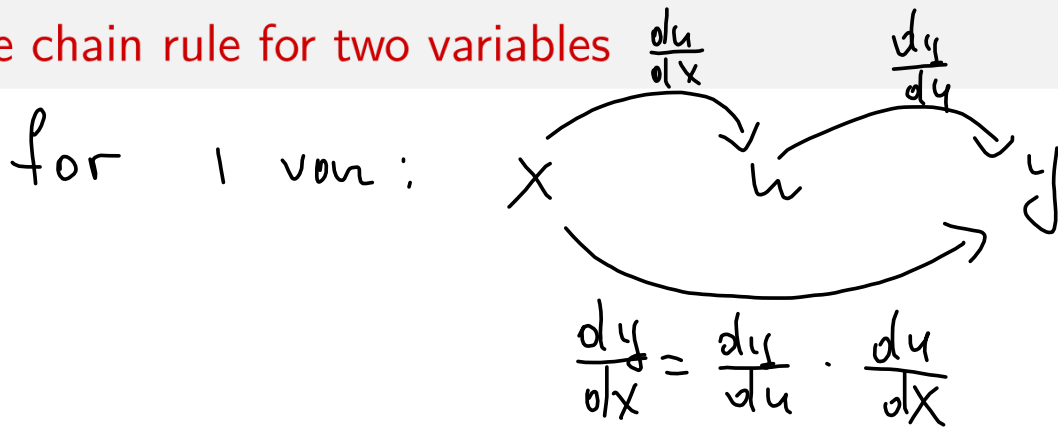
$$D(t) = 200 - 10(10 + 0.5t)^2 + 20(10 + 0.5t)(12.8 + 0.2t^2),$$

then take the derivative with respect to t ; or

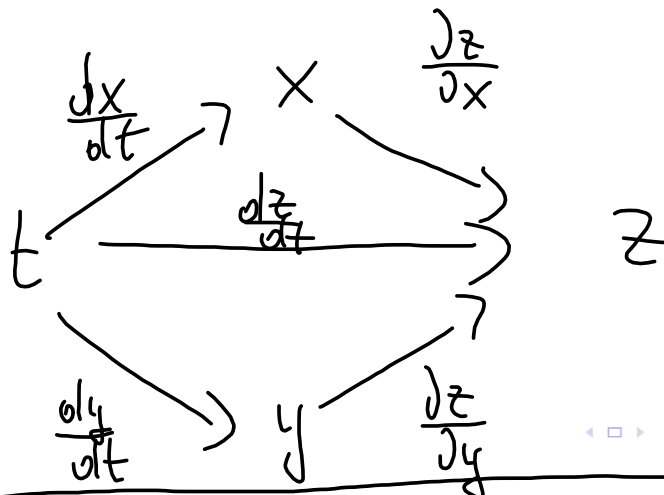
2. We can use what is called the chain rule for functions of two variables.



The chain rule for two variables



2 vars:



So

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$