McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

7.2: The chain rule

Angelica Babei

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Lecture 31

Higher-order partial derivatives

$$\begin{tikzcd}
 & & f_{xx} \\
 & f_x \arrow[ur] \arrow[r] & f_{xy} \\
f(x,y) \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & \\
 & f_y \arrow[r] \arrow[dr] & f_{yx} \\
 & & f_{yy}
\end{tikzcd}$$

These (e.g. f_{xy}, f_{yx}) are called "mixed 2nd order partials"

In general (this course) they are equal $f_{xy} = f_{yx}$.

It doesn't matter if you first differentiate w.r.t y, then w.r.t. x, or first w.r.t x then y.

This philosophy holds for higher order partials, e.g. $\underbrace{f_{xyy}} = \underbrace{f_{yxy}} = \underbrace{f_{yyx}}$

But, $f_{xyy} \neq f_{yxx}$

Another example

$$f(x,y) = (y+2)\ln(x^2+y^3)$$

Evaluate the second order partial derivatives at $(1,0)$. $(x=1, y=0)$

$$f_x = \frac{\partial}{\partial x}\left((y+2)\ln(x^2+y^3)\right) = (y+2)\frac{\partial}{\partial x}\left(\ln(x^2+y^3)\right)$$

$$= (y+2)\cdot\frac{\frac{\partial}{\partial x}(x^2+y^3)}{x^2+y^3} = \frac{(y+2)\cdot 2x}{x^2+y^3}$$

$$f_y = \frac{\partial}{\partial y}\left((y+2)\cdot\ln(x^2+y^3)\right) = \text{(use the product rule)}$$

$$= (y+2)\frac{\partial}{\partial y}\left(\ln(x^2+y^3)\right) + \ln(x^2+y^3)\cdot\frac{\partial}{\partial y}(y+2)$$

$$= (y+2)\cdot\frac{\frac{\partial}{\partial y}(x^2+y^3)}{x^2+y^3} + \ln(x^2+y^3)\cdot 1$$

$$= \frac{(y+2)\cdot 3y^2}{x^2+y^3} + \ln(x^2+y^3)$$

Example (cont.) $f_x = \frac{(y+2)\cdot 2x}{x^2+y^3}$ → Both have x (and y), so use quotient rule for f_{xx} and f_{xy}.

$$f_{xx} = \frac{LO\cdot dHI - HI\cdot dLO}{(x^2+y^3)^2}$$

$$= \frac{(x^2+y^3)\cdot \frac{\partial}{\partial x}\left((y+2)\cdot 2x\right) - (y+2)\cdot 2x \cdot \frac{\partial}{\partial x}(x^2+y^3)}{(x^2+y^3)^2}$$

$$= \frac{(x^2+y^3)\cdot (y+2)\cdot 2 - (y+2)\cdot 2x \cdot 2x}{(x^2+y^3)^2}$$

$= \frac{\partial}{\partial x}(2x)$

$$f_{xy} = (f_{yx}) = \frac{(x^2+y^3)\cdot \frac{\partial}{\partial y}\left((y+2)\cdot 2x\right) - (y+2)\cdot 2x \cdot \frac{\partial}{\partial y}(x^2+y^3)}{(x^2+y^3)^2}$$

$$= \frac{(x^2+y^3)\cdot 1 \cdot 2x - (y+2)\cdot 2x \cdot 3y^2}{(x^2+y^3)^2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{(y+2)\cdot 3y^2}{x^2+y^3} + \ln(x^2+y^3)\right)$$

$$= \frac{(x^2+y^3)\cdot \frac{\partial}{\partial y}\left((y+2)\cdot 3y^2\right) - (y+2)\cdot 3y^2 \cdot \frac{\partial}{\partial y}(x^2+y^3)}{(x^2+y^3)^2} + \frac{\frac{\partial}{\partial y}(x^2+y^3)}{x^2+y^3}$$

$$= \frac{(x^2+y^3)\cdot (9y^2+12y) - (y+2)\cdot 3y^2\cdot 3y^2}{(x^2+y^3)^2} + \frac{3y^2}{x^2+y^3}$$

At $(1,0)$, $f_{xx}(1,0) = \frac{2(1^2+0^3)\cdot(0+2) - 2\cdot 1\cdot 2\cdot 1\cdot (0+2)}{(1^2+0^3)^2}$

$= -4$

$$f_{xy}(1,0) = \frac{2\cdot 1\cdot (1^2+0^3)\cdot 1 - 2\cdot 1\cdot (0+2)\cdot 3\cdot 0^2}{(1^2+0^3)^2}$$

$= 2$

$f_{yy}(1,0) = 0$

Additional practice problems for basics of functions of 2 variables:
Exercises 1,3,5,7,9, 17, 19, 21, 23, 25, 27, 29, 31, 35 in Section 7.1
Additional practice problems for finding partial derivatives:
Exercises 1, 3, 7, 11, 13, 15, 17, 21, 23, 25, 27, 29, 31, 33 in Section 7.2

When the two variables x and y depend on a third one

Recall the demand function

$$D(x, y) = 200 - 10x^2 + 20xy$$

$x =$ price of the product, $y =$ price of the competing product.
Suppose that it is estimated that t months from now, the prices will be

$$x(t) = 10 + 0.5t, \qquad y(t) = 12.8 + 0.2t^2.$$

At what rate will the demand change with respect to time 4 months from now?

2 ways to answer the question:

1. We can rewrite $D(x, y)$ as a function of t, by substituting $x(t) = 10 + 0.5t$ and $y(t) = 12.8 + 0.2t^2$ in the expression for $D(x, y)$:

$$D(t) = 200 - 10(10 + 0.5t)^2 + 20(10 + 0.5t)(12.8 + 0.2t^2),$$

then take the derivative with respect to t; or

2. We can use what is called the chain rule for functions of two variables.

The chain rule for two variables

for 1 var:

$$\begin{tikzcd} x \arrow[r, "\frac{du}{dx}"] \arrow[rr, bend right] & u \arrow[r, "\frac{dy}{du}"] & y \end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2 vars:

$$\begin{tikzcd} & x \arrow[dr, "\frac{\partial z}{\partial x}"] & \\ t \arrow[ur, "\frac{dx}{dt}"] \arrow[rr, "\frac{dz}{dt}"] \arrow[dr, "\frac{dy}{dt}"'] & & z \\ & y \arrow[ur, "\frac{\partial z}{\partial y}"'] & \end{tikzcd}$$

So $\boxed{\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}}$