FACULTY OF SCIENCE
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7.2: The chain rule and marginal analysis

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Lecture 31

Example of the chain rule for 2 variables

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

$$D(x, y) = 200 - 10x^2 + 20xy$$

Suppose that it is estimated that t months from now, the prices will be $x(t) = 10 + 0.5t$ and $y(t) = 12.8 + 0.2t^2$. At what rate will the demand change with respect to time 4 months from now?

Solution. We calculated $\frac{\partial z}{\partial x} = -20x + 20y$ and $\frac{\partial z}{\partial y} = 20x$.

We can evaluate the rate of change of D with respect to t by using the chain rule:

Evaluate derivatives at $t = 4$, and $x(4)$, and $y(4)$.

$$\begin{tikzcd} & x \arrow[dr, "\frac{\partial z}{\partial x}"] & \\ t \arrow[ur, "\frac{dx}{dt}"] \arrow[dr, "\frac{dy}{dt}"'] & & z = D(x,y) \\ & y \arrow[ur, "\frac{\partial z}{\partial y}"'] & \end{tikzcd}$$

① $x(4) = 10 + 0.5 * 4 = 12$ $\quad y(4) = 12.8 + 0.2 * 4^2 = 16$

$\frac{\partial z}{\partial x} = -20x + 20y$, at $t = 4$,

$$\left.\frac{\partial z}{\partial x}\right|_{t=4} = -20 \cdot 12 + 20 \cdot 16 = 20(16 - 12) = 20 \cdot 4 = 80$$

$\frac{\partial z}{\partial y} = 20x$, at $t = 4$

$$\left.\frac{\partial z}{\partial y}\right|_{t=4} = 20 \cdot 12 = 240$$

② $\frac{dx}{dt} = 0.5$ $\quad \frac{dy}{dt} = 2 * 0.2t = 0.4t$

At $t = 4$, $\left.\frac{dx}{dt}\right|_{t=4} = 0.5$, $\left.\frac{dy}{dt}\right|_{t=4} = 0.4 * 4 = 1.6$

Finally, by the chain rule (in 2 vars),

$$\left.\frac{dz}{dt}\right|_{t=4} = \left.\frac{\partial z}{\partial x}\right|_{t=4} \cdot \left.\frac{dx}{dt}\right|_{t=4} + \left.\frac{\partial z}{\partial y}\right|_{t=4} \cdot \left.\frac{dy}{dt}\right|_{t=4}$$

$$= 80 * 0.5 + 240 * 1.6$$

$$= 40 + 384 = 424$$

(units / month)

Take a minute to digest!

Take a minute to jot down what you can remember from this example. Can you enumerate all the steps needed to calculate the rate of change of $f(x(t), y(t))$ with respect to a third variable t, at a specific point $t = t_0$?

Another example

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

$$f(x,y) = x^2 y, \quad x = 3t+1, \quad y = t^2 - 1$$

Find $\frac{df}{dt}$ at $t = 1$.

Rewrite chain rule: $\frac{df}{dt} = f_x \cdot \underbrace{x'(t)} + f_y \cdot \underbrace{y'(t)}$

2 sets of derivatives:

① $x'(t) = 3 \qquad y'(t) = 2t$

At $t = 1$, $x'(1) = 3$, $y'(1) = 2$

② $f_x = \frac{\partial}{\partial x}(x^2 y) = 2xy$

$f_y = \frac{\partial}{\partial y}(x^2 y) = x^2$

At $t=1$, $x(1) = 4$, $y(1) = 0$

So at $t=1$, $f_x = 2 \cdot 4 \cdot 0 = 0$

$f_y = 4^2 = 16$

Finally, $\frac{df}{dt} = 0 * 3 + 16 * 2 = 32$

Marginal analysis in 2 variables

Suppose we have $Q(x, y)$ and we need to approximate the change in $Q(x, y)$ as we vary x slightly. For example, how do $Q(x+1, y)$ and $Q(x, y+1)$ compare to $Q(x, y)$?

Answer: Recall marginal analysis in 1 variable, use partials instead of regular derivatives.

- for the x-variable: the approximate change in $Q(x,y)$ as we increase x by 1 is $Q_x(x,y)$

  So $Q(x+1,y) \approx Q(x,y) + Q_x(x,y)$

- for the y-variable: the change in $Q(x,y)$ as we increase y by 1 is approx. $Q_y(x,y)$

  So $Q(x,y+1) \approx Q(x,y) + Q_y(x,y)$

Simple example

Let $Q(x, y) = x^2 - y^2$. How much does $Q(x, y)$ approximately change at $(6, 5)$ if we increase x by 1?

$$Q(7,5) \approx Q(6,5) + Q_x(6,5)$$

so $Q(7,5) - Q(6,5) \approx Q_x(6,5)$

$$Q_x(x,y) = \frac{\partial}{\partial x}(x^2 - y^2) = 2x$$

$Q_x(6,5) = 2 \cdot 6 = 12$ (change is approx. 12)

Actual change: $Q(7,5) - Q(6,5) =$

$$= (7^2 - 5^2) - (6^2 - 5^2) = 49 - \cancel{25} - 36 + \cancel{25}$$
$$= 13$$

The general case

small

Generally, if we vary x by Δx and y by Δy, we have the formula

$$Q(x + \Delta x, y + \Delta y) \approx Q_x(x, y)\Delta x + Q_y(x, y)\Delta y + Q(x, y)$$

Example. If $Q(x, y) = e^{-(x^2+y^2)}$, approximate $Q(1.25, 0.25)$.

Another example

Example. For a function $f(x, y)$, you are given the following information:

$$f(5, -12) = 24, \quad f_x(5, -12) = -4, \quad f_y(5, -12) = 8.$$

Use this data to estimate $f(5.75, -12.5)$.

Soln: Pick values of $\underbrace{x, y}_{\text{know the function}}$, $\overbrace{\Delta x, \Delta y}^{\text{small}}$

Here, pick $x = 5$, $y = -12$

pick $\Delta x = 5.75 - 5 = 0.75$

$\Delta y = -12.5 - (-12) = -0.5$

So $f(5.75, -12.5) \approx f_x \cdot \Delta x + f_y \cdot \Delta y + f(x, y)$

$= -4 \cdot 0.75 + 8 \cdot (-0.5) + 24$

$= 17.$