McMaster University

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7.3: Optimization in 2 variables

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Lecture 32

Marginal analysis continued

Generally, if we vary x by Δx and y by Δy, we have the formula

$$Q(x+\Delta x, y+\Delta y) \approx Q_x(x,y)\Delta x + Q_y(x,y)\Delta y + Q(x,y)$$

Example. If $Q(x,y) = e^{-(x^2+y^2)}$ and $e \approx 2.71$, approximate $Q(1.25, 0.25)$.

Pick $x, y, \Delta x, \Delta y$. Let $x = 1$ so $\Delta x = 0.25$
$y = 0$ so $\Delta y = 0.25$

Need $Q(1,0)$ and the partials

$Q(1,0) = e^{-(1^2+0^2)} = e^{-1} \approx \frac{1}{2.71} = \frac{100}{271}$

$Q_x(x,y) = \frac{\partial}{\partial x}\left(e^{-x^2-y^2}\right) = e^{-x^2-y^2} \cdot \frac{\partial}{\partial x}(-x^2-y^2)$

$= -2x e^{-x^2-y^2}$

At $(1,0)$, $Q_x(1,0) = -2e^{-1}$

Example continued

$$Q_y(x,y) = \frac{\partial}{\partial y}\left(e^{-x^2-y^2}\right) = e^{-x^2-y^2} \cdot \frac{\partial}{\partial y}\left(-x^2-y^2\right)$$

$$= -2y\, e^{-x^2-y^2}$$

$$(1,0): \quad Q_y(1,0) = 0$$

$$\text{So, } Q(1.25, 0.25) \approx -2e^{-1} \cdot 0.25 + \cancelto{0}{0 \cdot 0.25} + e^{-1}$$

$$= \frac{1}{2} e^{-1} \approx \frac{50}{271} \approx 0.185$$

Actual value $Q(1.25, 0.25) \approx 0.197$

Another formulation of marginal analysis in 2 variables

Suppose z is a function of x and y. If Δx denotes a small change in x and Δy denotes a small change in y, the corresponding change in z, denoted by Δz, is approximated by

$$\underbrace{\Delta z}_{Q(x+\Delta x, y+\Delta y) - Q(x,y)} \approx \underbrace{\frac{\partial z}{\partial x}}_{Q_x(x,y)} \Delta x + \underbrace{\frac{\partial z}{\partial y}}_{Q_y(x,y)} \Delta y.$$

Additional practice for the chain rule: ex. 35, 37, 39, 63 in Chapter 7.2
Additional practice for marginal analysis: ex 47, 49, 51, 53, 61, 65, 67. 69, 77, 81 in Chapter 7.2

Optimization

For functions of one variable $f(x)$, we used the first and second-order derivative to find small and large values, with applications such as minimizing cost, maximizing profit etc.

The same philosophy works for functions in 2 variables, but now we involve the first and second-order partial derivatives.

We call the point (a, b) a **relative maximum** of the function $f(x, y)$ if

$$f(a, b) \geq f(x, y)$$

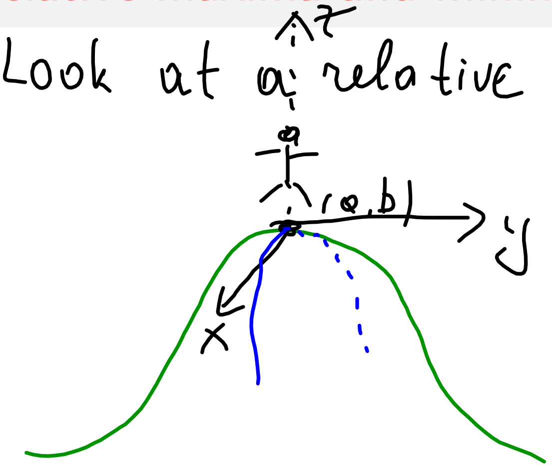
for all (x, y) in some disc around (a, b) (so it's a peak).
We call the point (a, b) a **relative minimum** of the function $f(x, y)$ if

$$f(a, b) \leq f(x, y)$$

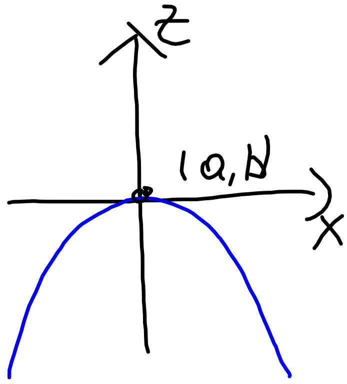
for all (x, y) in some disc around (a, b) (so it's a valley).
Relative maxima and relative minima are also called **relative extrema**.

Relative maxima and minima, and partial derivatives

Look at a relative max



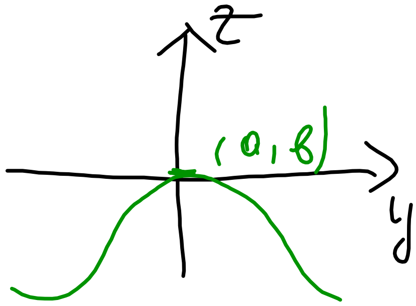
Slice by plane perpendicular to the y-axis



At (a,b) (the peak), the derivative of z with respect to x at (a,b)? It is zero!

So $\frac{\partial z}{\partial x} = f_x(x,y) = 0$

Next, we slice by a plane perpendicular to the x-axis (so x doesn't change)



Rate of change of z w.r.t. y at (a,b) is 0,

so $\frac{\partial z}{\partial y} = f_y(x,y) = 0$

Critical points

The pictures in the previous slide showed that at a relative maximum (a, b) (and, similarly, at a relative minimum), the partial derivatives $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Definition. A point (a, b) in the domain of $f(x, y)$ for which both partials $f_x(a, b) = 0$ and $f_y(a, b) = 0$ is called a **critical point**.

Good news. Relative extrema can only occur at critical points.

Examples: finding critical points

Crit. pts: $f_x(x,y) = 0 = f_y(x,y)$

Find the critical points of

$$f(x,y) = 2x^2 - 3y^2$$

① Find f_x, f_y

$f_x(x,y) = 4x$

$f_y(x,y) = -6y$

② Solve for x and y where both $f_x(x,y) = 0$ AND $f_y(x,y) = 0$.

Solve

$4x = 0$ and $-6y = 0$

So $x = 0$ and $y = 0$

So $(0,0)$ is the only critical pt.

Example II

Find the critical points of

$$f(x,y) = x^2 + 2y^2 - xy + 14y$$

(1) $f_x(x,y) = 2x - y$ $\quad f_y(x,y) = 4y - x + 14$

(2) $\underbrace{2x - y = 0}_{y = 2x}$ AND $4y - x + 14 = 0$

plug in $y = 2x$

$7x + 14 = 0$

$x = -2$

$y = 2(-2) = -4$

So $(-2,-4)$ is the only critical pt.

Example III

Find the critical points of

$$f(x,y) = x^2 - 6xy - 2y^3$$

① $f_x(x,y) = 2x - 6y$ $f_y(x,y) = -6x - 6y^2$

② Solve for x, y

$2x - 6y = 0$ $-6x - 6y^2 = 0$

$\underbrace{x - 3y = 0}$ $x + y^2 = 0$

$x = 3y$ plug in $x = 3y$

$3y + y^2 = 0$

$y(3+y) = 0$

So $y = 0$ or $y = -3$

So $y = 0$ gives $x = 3 \cdot 0 = 0$

$y = -3$ gives $x = 3 \cdot (-3) = -9$

2 critical points: $(0,0)$ and $(-9,-3)$

Take a minute to digest!

Take a minute to jot down what you can remember from these examples. Can you enumerate the steps we did to find critical points?
In addition, finding critical points requires one to solve two simultaneous equations. If you had any trouble solving the ones in these examples, make sure to make a note to review solving equations.