Sect2_Lect19-FTC

Saturday, February 25, 2023 12:00 PM

1MM3Lect...

McMaster University

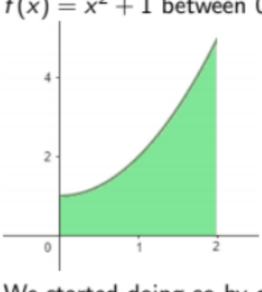
FACULTY OF SCIENCE
Department Of Mathematics & Statistics

Section 5.3: The area under a curve and the Fundamental Theorem of Calculus

MATH 1MM3 Winter 2023
Lecture 19

Section 5.3: The area under a curve

Last time, we were trying to approximate the area under the graph $f(x) = x^2 + 1$ between $0 \le x \le 2$.



We started doing so by dividing it into 2 pieces, and approximating their area by that of rectangles.

Approximating the area by smaller pieces (2 pieces)

Area of rectangles

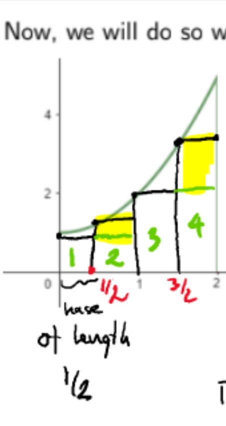
$= \boxed{1} \quad 1 \cdot f(0) = 1 \cdot (0^2+1) = 1$

$+ \boxed{2} \quad 1 \cdot f(1) = 1 \cdot (1^2+1) = 2$

total area = 3.

Approximating the area by smaller pieces (4 pieces)

Now, we will do so with 4: Area of rectangles.



$\boxed{1}: \frac{1}{2} \cdot f(0) = \frac{1}{2} \cdot (0^2+1) = \frac{1}{2}$

$\boxed{2}: \frac{1}{2} \cdot f(\frac{1}{2}) = \frac{1}{2}\left[(\frac{1}{2})^2+1\right] = \frac{5}{8}$

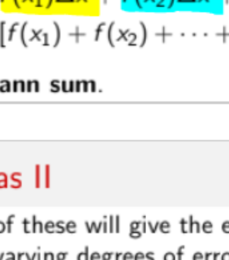
$\boxed{3}: \frac{1}{2} \cdot f(1) = \frac{1}{2}\left[1^2+1\right] = 1$

$\boxed{4} = \frac{1}{2} \cdot f(\frac{3}{2}) = \frac{1}{2}\left[(\frac{3}{2})^2+1\right] = \frac{13}{8}$

Total: $\frac{1}{2} + \frac{5}{8} + 1 + \frac{13}{8} = 3.75$

The overarching ideas I

Main idea I: We can get better and better estimates by making more and more subdivisions.



To find the area under graph of $f(x) \ge 0$ between $a \le x \le b$, let $n = \#$ subdivisions, $\Delta x = \frac{b-a}{n}$ the length of the base, and x_j where $j = 1, 2, \ldots, n$ the lower points on the recangles.
Then the new approximation will be

$$\text{Area}_{\text{approx}} = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$
$$= [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

This is called a **Riemann sum**.

The overarching ideas II

Main idea II: None of these will give the exact area, but just approximations with varying degrees of error. The exact area is obtained when we take the limit as the number of subdivisions $n \to \infty$.
Given a Riemann sum

$$\text{Area}_{\text{approx}} = [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x$$

taking the limit at $n \to \infty$ gives the exact area

$$\text{Area} = \lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x.$$

We will have a special notation for this limit:
Defn: The **definite integral** of $f(x)$ on the interval $a \le x \le b$ is the limit of the Riemann sums as $n \to \infty$, and written as

$$\int_a^b f(x)dx$$

integral notation → $\int$; upper bound → b; lower bound → a; integrand → $f(x)$

lim Riemann sum = integral

$\Delta x \longrightarrow dx$.

The fundamental theorem of calculus

We just saw that the area under the curve is written as

$$\text{Area} = \int_a^b f(x)dx$$

The way to calculate it using an antiderivative is via the **Fundamental Theorem of Calculus.**

Theorem (The Fundamental Theorem of Calculus)

If $f(x)$ is continuous on $a \le x \le b$, and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Back to our example $\int_0^2 (x^2+1)dx$

$$\text{Area} = \int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Our candy wrapper had the area of the region under $f(x) = x^2 + 1$ between $0 \le x \le b$. The antiderivative of $f(x)$ is of the form

$$F(x) = \int (x^2+1)dx = \frac{1}{3}x^3 + x + C, \quad \text{so the area is}$$

$$\text{Area} = \left(\frac{1}{3}x^3 + x + C\right)\Big|_0^2 = \left(\frac{1}{3}\cdot 2^3 + 2 + C\right) - \left(\frac{1}{3}\cdot 0^3 + 0 + C\right)$$
$$= \frac{8}{3} + 2 + C - 0 - 0 - C = \frac{14}{3}$$

Note. The constant C always cancels, just as above. So we can just take the antiderivative to have constant $C = 0$, e.g. above take $F(x) = \frac{1}{3}x^3 + x$.

Examples I

Evaluate the definite integral

$$\int_1^4 2\sqrt{u}du$$

Solution. $\int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{1/2+1}}{1/2+1} + C = \frac{2}{3}u^{3/2} + C$

$\int_1^4 2\sqrt{u}\,du = 2\cdot\frac{2}{3}u^{3/2}\Big|_1^4$

$= \frac{4}{3}\cdot 4^{3/2} - \frac{4}{3}\cdot 1^{3/2}$

$= \frac{4}{3}\cdot(\sqrt{4})^3 - \frac{4}{3}\cdot 1$

$= \frac{4}{3}\cdot 2^3 - \frac{4}{3} = \boxed{\frac{28}{3}}$

Examples II

Evaluate the definite integral

$$\int_{-1}^2 \left(\frac{1}{e^x} - \frac{1}{e^{-x}}\right)dx$$

Solution.

$\int_1^2 \left(\frac{1}{e^x} - \frac{1}{e^{-x}}\right)dx = \int_1^2 (e^{-x} - e^x)dx$

$= \left(\frac{e^{-x}}{-1} - e^x\right)\Big|_{-1}^2$

$= (-e^{-x} - e^x)\Big|_{-1}^2$

$= (-e^{-2} - e^2) - (-e^1 - e^{-1})$

$= -\frac{1}{e^2} - e^2 + e + \frac{1}{e}$

Examples III

It is estimated that at a country fair at t hours after the gates open at 9 am, visitors will be entering at the rate of $175t - 25t^2$ people per hour. How many people will enter between 11 am and 1 pm?

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Soln. Let $N(t) = \#$ people entering the fair t hours after 9 am. We know that the rate that people enter the fair is

$$N'(t) = 175t - 25t^2.$$

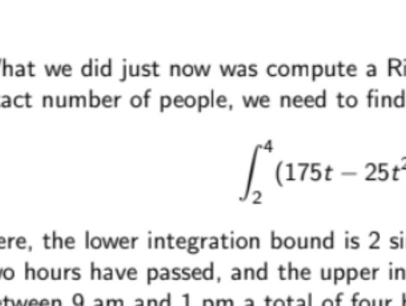
We could approximate the answer by dividing the time interval between 11 and 1 into two one-hour subintervals, between 11 and 12, and between 12 and 1.

Examples III (cont.) $N'(2)$

In the first case, 11 am would give $t = 2$ (2 hours after 9 am), and approximately $175 \cdot 2 - 25 \cdot 2^2 = 350 - 100 = 250$ people enter the fair between 11 and 12.

In the second case, $t = 3$, and approximately $175 \cdot 3 - 25 \cdot 3^2 = 525 - 225 = 300$ people enter the fair between 12 and 1. ← $N'(3)$

In total, this is approximately 550 people.



Examples III (cont.)

What we did just now was compute a Riemann sum! To get the exact number of people, we need to find

$$\int_2^4 (175t - 25t^2).$$

Here, the lower integration bound is 2 since between 9 and 11 am two hours have passed, and the upper integration bound is 4 since between 9 am and 1 pm a total of four hours have passed.

Examples III (cont.)

From 11 am to 1 pm
$2 \le t \le 4$

$$\int_2^4 (175t - 25t^2) = \left(\frac{175t^2}{2} - \frac{25t^3}{3}\right)\Big|_2^4$$
$$= \left(\frac{175\cdot 4^2}{2} - \frac{25\cdot 4^3}{3}\right) - \left(\frac{175\cdot 2^2}{2} - \frac{25\cdot 2^3}{3}\right)$$
$$= \frac{2600}{3} - \frac{850}{3} = \frac{1750}{3} \approx 583 \text{ (people)}.$$

← $N(4) - N(2)$

Note. To make the calculations easier, it's usually best to calculate the number in each parenthesis separately, then subtract, as above.