McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

7.3: Optimization on closed, bounded regions // 7.5: The method of Lagrange multipliers

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Lecture 35

Example from last time

Example. Find possible extrema of $f(x, y) = x^2 - 2xy + 2y^2$ on the triangle with vertices $(0, 0), (1, 0), (1, 1)$.

Solution.

On the segment $y = 0, 0 \leq x \leq 1$, we found possible extrema at $(0, 0)$ and $(1, 0)$.

On the segment $x = 1, 0 \leq y \leq 1$ we found possible extrema at $(1, 0), (1, 1/2)$ and $(1, 1)$.

Next, find possible extrema on the segment $y = x, 0 \leq x \leq 1$.

Recall the Extreme Value Property in 2 variables

Theorem. If $f(x, y)$ is continuous (i.e. it has no ruptures in its graph) on a closed, bounded region R, it attains both an absolute max and an absolute min on R. They will either occur at critical points (inside R), or on the boundary of R.

We have 3 steps for finding absolute extrema:

1. Find the critical points inside R;
2. Find the possible extrema on the boundary of R;
3. Evaluate the function at the points in (1) and (2), and compare the values.

Example

Determine the absolute extrema of

$$f(x, y) = 4xy - 8x - 4y + 5$$

on the triangle region with vertices $(0, 0), (2, 0)$ and $(0, 3)$.

Example (cont.)

Example (cont.)

Example (cont.)

Additional practice for finding absolute minima and maxima on closed, bounded regions: Exercises 23, 25, 37, 43 in Chapter 7.3.

7.5: Constrained optimization; Lagrange multipliers

Motivating example. The output of a factory is given by $f(x, y) = 42x^{1/5}y^{4/5}$, where $x =$ thousands of dollars spent on labor, and $y =$ thousands of dollars spent on parts. What is the maximum output if the budget is \$197000 dollars?

The method of Lagrange multipliers

Main idea: To optimize $f(x, y)$ with respect to the constraint $g(x, y) = k$, we find the critical points of a new function $F(x, y, \lambda)$ of one more variable λ, where

$$F(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - k].$$

The method of Lagrange multipliers (cont.)

$$F(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - k].$$

The partials F_x, F_y and F_λ are

If $f(x, y)$ has a smallest or largest value subject to the constrant $g(x, y) = k$, it will occur at one of the critical points of $F(x, y, \lambda)$. So we need to solve for x and y the system of equations

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x, y) &= k \end{aligned}$$

Back to our example

Optimize the function

$$f(x, y) = 42x^{1/5}y^{4/5}$$

with respect to the constraint $1000x + 1000y = 197000$.

Example (cont.)

Example (cont.)