FACULTY OF SCIENCE
Department Of Mathematics & Statistics

7.3: Optimization in 2 variables (cont.)

Angelica Babei

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Lecture 32

Classifying critical points

Recall: (a,b) is a critical point if both $f_x(a,b)=0$ and $f_y(a,b)=0$.

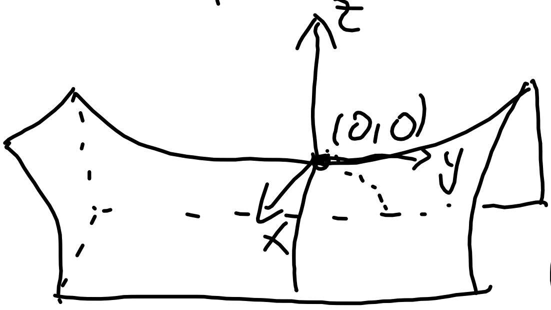
Bad news: Not all critical points are relative extrema. We will need more information to classify critical points as maxima, minima, or neither.

e.g: $f(x,y) = y^2 - x^2$, $f_x(x,y) = -2x$

$(0,0)$ only critical pt. $f_y(x,y) = 2y$

Graph:

saddle



$(0,0)$ neither min nor max (saddle point)

The second partial derivative test

Let $f(x, y)$ with partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$, and let

$$D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - [f_{xy}(x, y)]^2 .$$

If at a critical point (a, b), all the first and second partial derivatives are defined, then

- if $D(a, b) < 0$, the critical point is a saddle point;
- if $D(a, b) > 0$ and $f_{xx}(a, b) < 0$ (or $f_{yy} < 0$)), the critical point is a relative maximum;
- if $D(a, b) > 0$ and $f_{xx}(a, b) > 0$ (or $f_{yy} > 0$)), the critical point is a relative minimum.

If $D(a, b) = 0$, the test is inconclusive.

Example I

In the previous example I, classify the critical points.

$$f(x, y) = 2x^2 - 3y^2$$

Solution. We have already found $f_x(x, y) = 4x$, $f_y(x, y) = -6\cancel{x}y$, and the only critical point is $(0, 0)$.

$$\left.\begin{aligned} f_{xx} &= 4 \\ f_{yy} &= -6 \\ f_{xy} &= 0 \end{aligned}\right\} \begin{aligned} &D(x,y) = 4\cdot(-6) - 0^2 = -24 \\ &\text{At } (0,0),\ D(0,0) = -24 < 0 \end{aligned}$$

So a saddle point!

Example III

In the previous example III, classify the critical points.

$$f(x,y) = x^2 - 6xy - 2y^3$$

Solution. We have already found $f_x(x,y) = 2x - 6y$, $f_y(x,y) = -6x - 6y^2$, and critical points are $(0,0)$ and $(-9,-3)$.

$$\left.\begin{array}{l} f_{xx} = 2 \\ f_{yy} = -12y \\ f_{xy} = -6 \end{array}\right\}$$

$D(x,y) = 2\cdot(-12y) - (-6)^2 = -24y - 36$

At $(0,0)$, $D(0,0) = -36 < 0$, saddle point

At $(-9,-3)$, $D(-9,-3) = -24\cdot(-3) - 36 = 36 > 0$

$f_{xx} = 2 > 0$, relative minimum

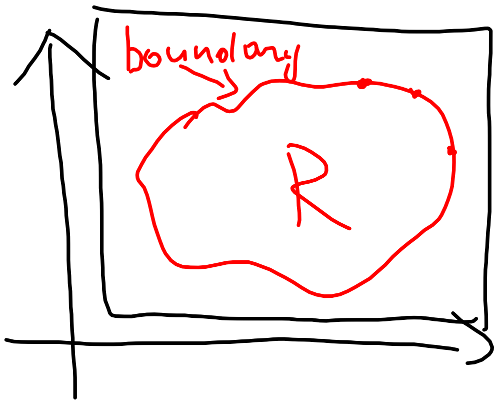
(Alternatively, $f_{yy}(-9,-3) = -12\cdot(-3) = 36 > 0$) so also rel. min

Additional practice for finding and classifying critical points:
1,3,5,7,9,11, 13, 15, 17, 19, 21 in Chapter 7.3

Smallest and largest values on a closed, bounded region

So far: relative (local) extrema.
Now: absolute maximum (largest value), and absolute minimum (smallest value).
As in the single variable case, absolute extrema are not guaranteed to exist. However, they do in some cases, such as when we examine the function on a closed, bounded region.

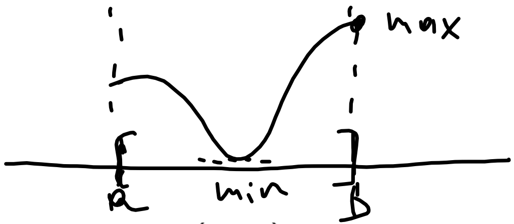
A region is bounded if it can be enclosed in some circle (or rectangle). Think "finite"



A region is closed if it contains its boundary.

The Extreme Value Property in 2 variables

Recall the Extreme value property in one variable: If $f(x)$ is continuous on a **closed** interval, it attains both an absolute minimum and an absolute maximum, which either occur at critical points inside the interval, or at the endpoints.



① Find crit. pts
② Evaluate at endpoints
③ Compare values

Theorem. If $f(x, y)$ is continuous (i.e. it has no ruptures in its graph) on a closed, bounded region R, it attains both an absolute max and an absolute min on R. They will either occur at critical points (inside R), or on the boundary of R.

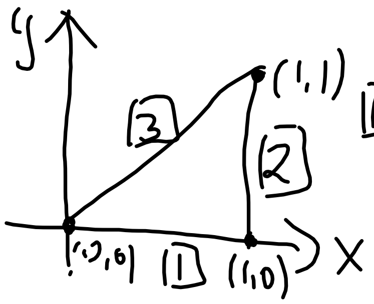
We have 3 steps for finding absolute extrema:

1. Find the critical points inside R; → straight forward
2. Find the possible extrema on the boundary of R; → ???
3. Evaluate the function at the points in (1) and (2), and compare the values. } easy

Extrema on the boundary

Main idea for Step 2: on the boundary, the function behaves like a function of a single variable. Divide the boundary into closed segments (on each segment the single variable function will be different), then use the Extreme value property in one variable.

Example. Find possible extrema of $f(x, y) = x^2 - 2xy + 2y^2$ on the triangle with vertices $(0, 0), (1, 0), (1, 1)$.



3 line segments:

$\boxed{1}$ $y = 0, \quad 0 \le x \le 1$

$\boxed{2}$ $x = 1, \quad 0 \le y \le 1$

$\boxed{3}$ $y = x, \quad 0 \le x \le 1$

Next, we will use the Extreme value property in 1 variable on each segment

$\boxed{1}$ $y = 0$, so $f(x,y)$ is

$f(x,0) = x^2 = u(x)$, fcn. of one variable

Extrema on this segment: at critical pts of $u(x)$, or endpoints.

$u'(x) = 2x$, solve $2x = 0$ so $x = 0$

Possible extrema: crit. pt. $(0,0)$

or endpoints

$(0,0), (1,0)$

Example (cont.)

on $\boxed{2}$, $x=1$, $0\le y\le 1$

there, $f(x,y)$ is

$$f(1,y)=1-2y+2y^2=v(y)$$

Extreme value prop. in 1 variable:

$v'(y)=-2+4y$, solve $-2+4y=0$

$y=\frac{1}{2}$

· Possible extrema: crit. pt $(1,\frac{1}{2})$

or endpoints $(1,0),(1,1)$.