McMaster University
FACULTY OF SCIENCE
Department Of Mathematics & Statistics

7.3: Optimization on closed, bounded regions // 7.5: The method of Lagrange multipliers

Angelica Babei

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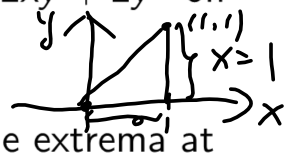
Lecture 35

Example from last time

Example. Find possible extrema of $f(x,y) = x^2 - 2xy + 2y^2$ on the triangle with vertices $(0,0), (1,0), (1,1)$.



Solution.

On the segment $y = 0, 0 \le x \le 1$, we found possible extrema at $(0,0)$ and $(1,0)$.

On the segment $x = 1, 0 \le y \le 1$ we found possible extrema at $(1,0), (1,1/2)$ and $(1,1)$.

Next, find possible extrema on the segment $y = x, 0 \le x \le 1$.

On this segment, $f(x,y)$ is

$$f(x,x) = x^2 - 2\cdot x\cdot x + 2x^2 = \underbrace{x^2 = w(x)}_{\text{fcn of one variable}}$$

To use the extreme value property in one variable, find critical pts of $w(x)$ on $0 \le x \le 1$

$w'(x) = 2x$, set $2x = 0$ so $x = 0$.

Possible extrema are $(0,0), (1,1)$ (just the endpoints).

In total, need to look at

$(0,0), (1,0), (1,1), (1, 1/2)$.

Recall the Extreme Value Property in 2 variables

defined everywhere in the domain.

Theorem. If $\underbrace{f(x,y) \text{ is continuous}}$ (i.e. it has no ruptures in its graph) on a closed, bounded region R, it attains both an absolute max and an absolute min on R. They will either occur at critical points (inside R), or on the boundary of R.

We have 3 steps for finding absolute extrema: (points or values)

1. Find the critical points inside R;
2. Find the possible extrema on the boundary of R; ⇝ just did
3. Evaluate the function at the points in (1) and (2), and compare the values.

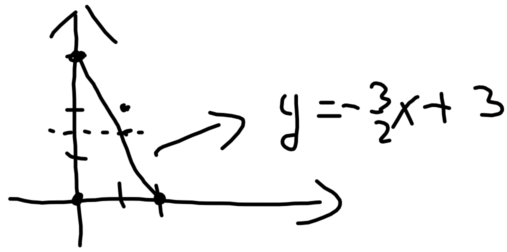
Example

Determine the absolute extrema of

$$f(x, y) = 4xy - 8x - 4y + 5$$

on the triangle region with vertices $(0, 0)$, $(2, 0)$ and $(0, 3)$.

$\boxed{0}$ Sketch region



$\boxed{1}$ Find f_x, f_y.

$$\left.\begin{aligned} f_x &= 4y - 8 \\ f_y &= 4x - 4 \end{aligned}\right\}$$ set them $= 0$

$4y - 8 = 0$
$4x - 4 = 0$ $\leadsto$ $y = 2$, $x = 1$

One critical point. $(1, 2)$ (Not in Δ !!)

To check: is it above or below

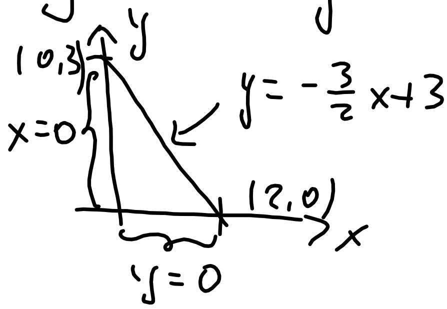
At $x = 1$, y-value on line $y = -\frac{3}{2}x + 3$ is $-\frac{3}{2} + 3 = 1.5$, so pt on line is $(1, 1.5)$. Our point is $(1, 2)$, so above

No critical points inside the triangle!

Example (cont.)

$f(x,y) = 4xy - 8x - 4y + 5$

$\boxed{2}$ Look at boundary:



3 line segments:

☆ $y = 0, \quad 0 \le x \le 2$

There, $f(x,y)$ is

$$f(x,0) = -8x + 5 = u(x)$$

Then $u'(x) = -8$, no critical pts of $u(x)$ on segment.

Points of interest: endpoints $(0,0), (2,0)$

☆ $x = 0, \quad 0 \le y \le 3$

There, $f(x,y)$ is

$$f(0,y) = -4y + 5 = v(y)$$

Then $v'(y) = -4$, no critical pts of $v(y)$ on the segment.

Points of interest: endpoints $(0,0), (0,3)$.

Example (cont.)

☆ $y = -\frac{3}{2}x + 3, \quad 0 \le x \le 2$

there, $f(x,y)$ is

$$f\left(x, -\frac{3}{2}x + 3\right) = 4x\left(-\frac{3}{2}x+3\right) - 8x$$
$$-4\left(-\frac{3}{2}x+3\right) + 5 = -6x^2 + 10x - 7 = w(x)$$

On this segment, critical pts of $w(x)$ are at $w'(x) = 0$:

$w'(x) = -12x + 10$, solve $-12x + 10 = 0$

so $x = \frac{5}{6} \rightsquigarrow y = -\frac{3}{2}\cdot\frac{5}{6} + 3 = 7/4$

It is inside $0 \le x \le 2$.

Points of interest: $(2,0), (0,3), \left(\frac{5}{6}, \frac{7}{4}\right)$

Example (cont.)

$\boxed{3}$ Compare values

(x,y)	$(0,0)$	$(2,0)$	$(0,3)$	$\left(\frac{5}{6},\frac{7}{4}\right)$
$f(x,y)$	5	-11	-7	$-17/6$

Largest smallest.

Absolute max is 5
min is -11.

Additional practice for finding absolute minima and maxima on closed, bounded regions: Exercises 23, 25, 37, 43 in Chapter 7.3.

7.5: Constrained optimization; Lagrange multipliers

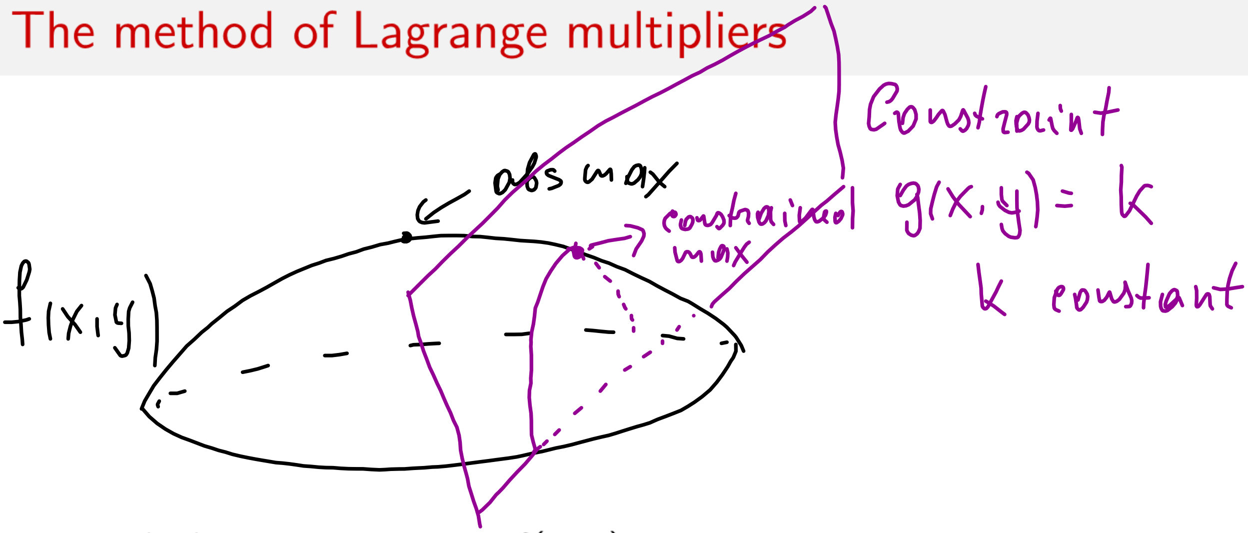
Motivating example. The output of a factory is given by $f(x, y) = 42x^{1/5}y^{4/5}$, where x = thousands of dollars spent on labor, and y = thousands of dollars spent on parts. What is the maximum output if the budget is \$197000 dollars?

We have a budget constraint:
this is given by the function

$$g(x,y) = 1000x + 1000y$$

The constraint is $g(x,y) = 197\,000$

The method of Lagrange multipliers



Main idea: To optimize $f(x, y)$ with respect to the constraint $g(x, y) = k$, we find the critical points of a new function $F(x, y, \lambda)$ of one more variable λ, where

$$F(x, y, \lambda) = f(x, y) - \lambda[g(x, y) - k].$$

The method of Lagrange multipliers (cont.)

$$F(x,y,\lambda) = f(x,y) - \lambda[g(x,y) - k].$$ crit. pts?

$= f(x,y) - \lambda g(x,y) + \lambda k$

The partials F_x, F_y and F_λ are

$$F_x = f_x - \lambda g_x + 0$$
$$F_y = f_y - \lambda g_y + 0$$
$$F_\lambda = 0 - g(x,y) + k$$

set $= 0$

If $f(x,y)$ has a smallest or largest value subject to the constrant $g(x,y) = k$, it will occur at one of the critical points of $F(x,y,\lambda)$. So we need to solve for x and y the system of equations

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x,y) &= k \end{aligned}\right\}$$

Back to our example

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$g(x,y) = k$$

Optimize the function

$$f(x,y) = 42x^{1/5}y^{4/5}$$

with respect to the constraint $1000x + 1000y = 197000$.

Soln: Constraint is $g(x,y) = 1000x + 1000y = 197000$

$$f_x = 42 \cdot \frac{1}{5} x^{-\frac{4}{5}} y^{\frac{4}{5}} \qquad g_x = 1000$$

$$f_y = 42 \cdot \frac{4}{5} x^{\frac{1}{5}} y^{-\frac{1}{5}} \qquad g_y = 1000$$

Solve the system:

$$\left.\begin{aligned} &\frac{42}{5} x^{-\frac{4}{5}} y^{\frac{4}{5}} = 1000\lambda \\ &\frac{42 \cdot 4}{5} x^{\frac{1}{5}} y^{-\frac{1}{5}} = 1000\lambda \\ &1000x + 1000y = 197000 \end{aligned}\right\}$$