McMaster University

FACULTY OF SCIENCE
Department Of Mathematics & Statistics

Review Part 1

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MATH 1MM3 Winter 2023
Lecture 37

- Please fill in the end of term course evaluations: **mcmaster.bluera.com/mcmaster**
- Exam covers all the material in the course; 30 questions, roughly 1/3 is on material since Midterm 3.
- The formula sheet covers present and future values, integration by parts, marginal analysis, and the second partials test
- Regular office hours this and next week: Wednesday 12:20-1:20, Thursday 1:30-2:30, and by appointment.
- Practice tests: there are two sample tests available on Childsmath. Final sample: do exercises 2, 3, 4, 5, 6, 7, 8, 13, 14, 15, 16, 17, 18, 19, 20, 23, 26, 27, 30. Final 0.1 sample: do exercises 1, 2, 3, 4, 6, 7, 8, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 29, 30.

Partial derivatives

$f_x(x, y) =$ rate of change of f with respect to x
$\quad = $ differentiate with respect to x, while treating y as a constant

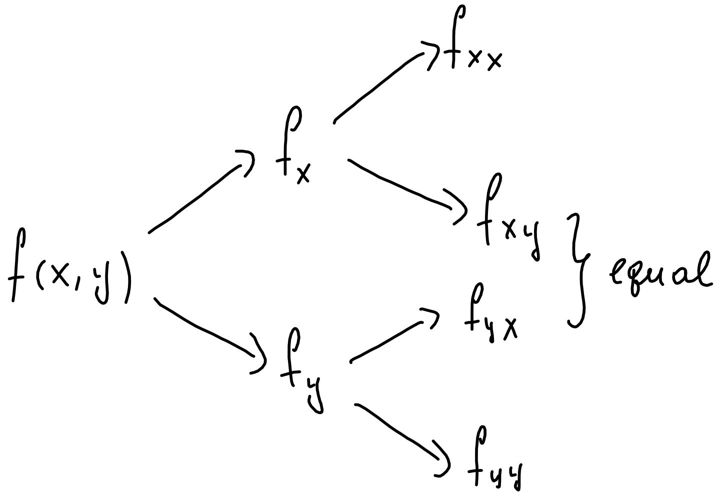
$f_y(x, y) =$ rate of change of f with respect to y
$\quad = $ differentiate with respect to y, while treating x as a constant

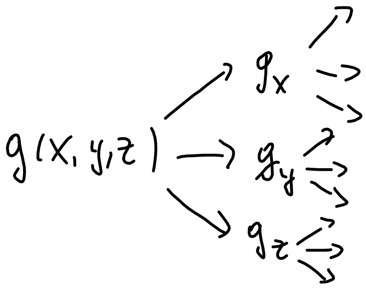
Example. $f(x, y) = e^{x^2 y}$

$$f_x = \frac{\partial}{\partial x}(x^2 y) \cdot e^{x^2 y} = 2xy e^{x^2 y}$$

$$f_y = \frac{\partial}{\partial y}(x^2 y) \cdot e^{x^2 y} = x^2 e^{x^2 y}$$

Higher order derivatives





e.g.: $f(x,y) = e^{x^2y}$ $\quad f_x = 2xy\,e^{x^2y}$ $\quad f_y = x^2 e^{x^2y}$

$$f_{xx} = \frac{\partial}{\partial x}\left(2xy\,e^{x^2y}\right) = \frac{\partial}{\partial x}(2xy)\cdot e^{x^2y} + \frac{\partial}{\partial x}\left(e^{x^2y}\right)\cdot 2xy$$

$$= 2y\,e^{x^2y} + 2xy\cdot e^{x^2y}\cdot 2xy$$

$$= 2y e^{x^2y}\left(1 + 2x^2y\right)$$

$$f_{yy} = \frac{\partial}{\partial y}\left(x^2 e^{x^2y}\right) = x^2\cdot\frac{\partial}{\partial y}\left(e^{x^2y}\right) = x^4 e^{x^2y}$$

$$f_{xy} = f_{yx} = \frac{\partial}{\partial x}\left(x^2 e^{x^2y}\right) = \frac{\partial}{\partial x}(x^2)\cdot e^{x^2y} + \overbrace{\frac{\partial}{\partial x}\left(e^{x^2y}\right)}^{f_x}\cdot x^2$$

$$= 2x e^{x^2y} + 2xy e^{x^2y}\cdot x^2$$

$$= 2x\,e^{x^2y}\left(1 + x^2y\right)$$

Critical points and classification

To find critical points, solve the system

$$\text{Both true} \begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

Example. Find the critical points of
$f(x, y) = (x-1)^2 + y^3 - 3y^2 - 9y + 5$

$$f_x = 2(x-1) = 0 \rightsquigarrow x = 1$$

$$f_y = 3y^2 - 6y - 9 = 0$$

$$y^2 - 2y - 3 = 0$$

$$(y+1)(y-3) = 0 \rightsquigarrow y = -1, 3$$

So 2 critical pts : $(1,-1)$
$(1,3)$

Classifying critical points

$$\boxed{D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - [f_{xy}(x,y)]^2 .}$$

- if $D(a,b) < 0$, the critical point is a saddle point;
- if $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ (or $f_{yy} < 0$)), the critical point is a relative maximum;
- if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ (or $f_{yy} > 0$)), the critical point is a relative minimum.

If $D(a,b) = 0$, the test is inconclusive.

Example. Classify the critical points of $f(x,y) = (x-1)^2 + y^3 - 3y^2 - 9y + 5$

$f_x = 2(x-1)$ $\quad f_y = 3y^2 - 6y - 9$ $\quad$ 2 crit pts: $(1,-1), (1,3)$

$f_{xx} = 2$

$f_{yy} = 6y - 6$

$f_{xy} = 0$

$D(1,-1) = 2\cdot(6\cdot(-1)-6) - 0^2 = -24 < 0$

$D(1,3) = 2\cdot(6\cdot 3 - 6) - 0^2 = 24 > 0$, $f_{xx} = 2 > 0$

$(1,-1)$: saddle pt.

$(1,3)$: relative/local minimum.

The chain rule $z = f(x,y)$, $x(t)$, $y(t)$

$$\begin{tikzcd} & x \arrow[dr, "\frac{\partial z}{\partial x}"] & \\ t \arrow[ur, "\frac{dx}{dt}"] \arrow[dr, "\frac{dy}{dt}"'] & & z \\ & y \arrow[ur, "\frac{\partial z}{\partial y}"'] & \end{tikzcd}$$

2 sets of derivatives:

① $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

② $\frac{dx}{dt}, \frac{dy}{dt}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

e.g: $z = \frac{3x}{y} = 3xy^{-1}$ $\quad x = t \quad y = t^2$

Find $\frac{dz}{dt}$ in terms of x, y, and t

$\frac{\partial z}{\partial x} = 3y^{-1}$ | $\frac{dx}{dt} = 1$

$\frac{\partial z}{\partial y} = 3x \cdot (-1) \cdot y^{-2} = -3xy^{-2}$ | $\frac{dy}{dt} = 2t$

Then $\frac{dz}{dt} = 3y^{-1} \cdot 1 - 3xy^{-2} \cdot 2t$

$$= \frac{3}{y} - \frac{6xt}{y^2}$$

② What is $\frac{dz}{dt}$ at $t=2$?

$x = t \qquad y = t^2$

At $t=2$:

$x = 2 \qquad y = 4$

So $\left.\frac{dz}{dt}\right|_{t=2} = \frac{3}{4} - \frac{6 \cdot \cancel{2} \cdot 2}{4^{\cancel{2}}} = -\frac{3}{4}$

Marginal analysis

Chain rule: $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$

Marginal analysis: $\Delta z \approx \frac{\partial z}{\partial x} \cdot \Delta x + \frac{\partial z}{\partial y} \cdot \Delta y$

formula $\rightsquigarrow$ what

$\delta z \approx \frac{\partial z}{\partial x} \cdot \delta x + \frac{\partial z}{\partial y} \cdot \delta y$

$\Delta z, \Delta x, \Delta y$ = change in z, x, y.

$\frac{dz}{dt}, \frac{dx}{dt}, \frac{dy}{dt}$ = change in z, x, y with respect to t.

Lagrange multipliers

If we need to optimize the function $f(x, y)$ with respect to the constraint $g(x, y) = k$, we solve for (x, y) the system

$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g(x, y) &= k \end{aligned}}$$

variables
x, y, λ

This method extends to 3 variables:
If we need to optimize the function $f(x, y, z)$ with respect to the constraint $g(x, y, z) = k$, we solve for (x, y, z) the system

$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x, y, z) &= k \end{aligned}}$$

variables
x, y, z, λ

Example

Solve for $x, y, z > 0$ the system

$$26y + 18z = \lambda yz \qquad \lambda = \frac{26y+18z}{yz}$$

$$26x + 18z = \lambda xz \qquad \lambda = \frac{26x+18z}{xz}$$

$$18x + 18y = \lambda xy \qquad \lambda = \frac{18x+18y}{xy}$$

$$xyz = 1053$$

$$\frac{26y+18z}{yz} = \frac{26x+18z}{xz}$$ mult both sides by xyz

$$(26y+18z)x = (26x+18z)y \rightsquigarrow \cancel{18}x\cancel{z} = \cancel{18}y\cancel{z}$$

$$\boxed{x=y}$$

$$\frac{26x+18z}{xz} = \frac{18x+18y}{xy}$$ mult by xyz

$$(26x+18z)y = (18x+18y)\cdot z$$

$$26xy + \cancel{18yz} = 18xz + \cancel{18yz}. \quad \text{But } x=y!$$

$$26x^2 = 18xz$$

$$\boxed{z = \frac{26}{18}x = \frac{13}{9}x} \quad \text{and} \quad \boxed{x=y}$$

Plug in $xyz = 1053$