McMaster University
FACULTY OF SCIENCE
Department Of Mathematics & Statistics

Review Part 2

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MATH 1MM3 Winter 2023
Lecture 38

Please fill in the end of term course evaluations:
mcmaster.bluera.com/mcmaster

Math Help Centre is open, in-person and online, during the exam period until Thurs. April 27th. Times are as usual: in-person from 2:30-6:30pm (14:30-18:30) Mon-Fri, and online 8:30am-2:30pm (8:30-14:30) in MS Teams.

$$\begin{tikzcd}
 & \text{1MM3} \arrow[dl] \arrow[dr] & \\
\text{Theory} \arrow[rr, bend right] & & \text{Applications}
\end{tikzcd}$$

Theory Topics

1. New types of functions

- exponential $f(x) = e^x$
- logarithmic $f(x) = \ln x$, $\log_b x$ etc.
- 2 or more variables: $f(x, y)$, $g(x, y, z)$

Concepts to review:

① Domain

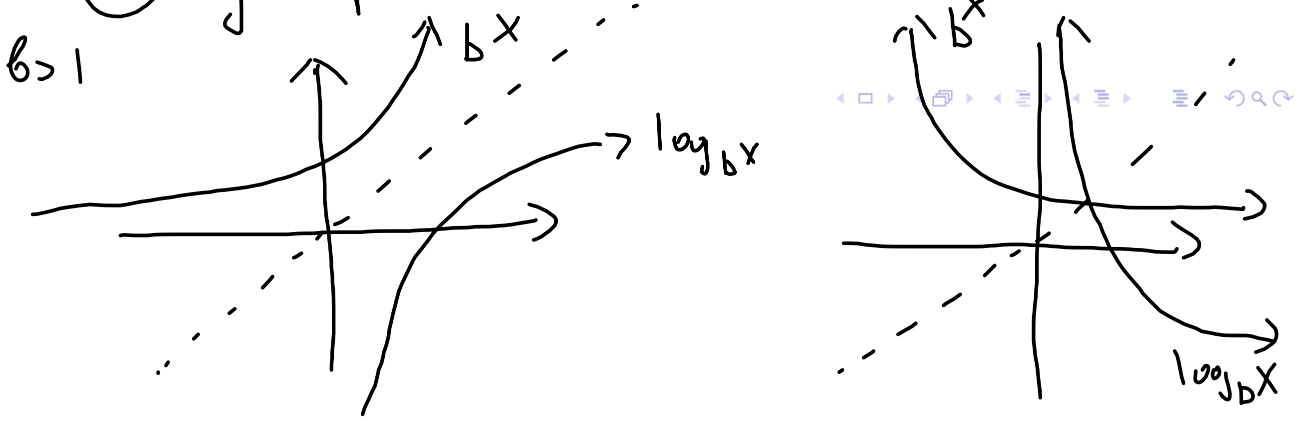
e^x : Domain $\mathbb{R}$

$\ln x$: $(0, +\infty)$

$f(x,y)$: Region in xy-plane

② graphs

$0 < b < 1$



$f(x,y)$: find level curves

Take heights $z = h$, solve $f(x,y) = h$

2. Differentiation

$$\left.\begin{aligned}(e^x)' &= e^x \\ (\ln x)' &= \frac{1}{x}\end{aligned}\right\} \text{ also for base } b \qquad \left.\begin{aligned}(e^{u(x)})' &= u'(x)e^{u(x)} \\ (\ln u(x))' &= \frac{u'(x)}{u(x)}\end{aligned}\right\} \text{ Chain rule.}$$

for $f(x, y)$: partial differentiation

Concepts to review:

1. The chain rule
2. Differential equations : in one variable only.

$$\frac{dy}{dx} = f(x) \overset{\text{integrate}}{\rightsquigarrow} y = \int f(x)\,dx$$

Separable $$\frac{dy}{dx} = \frac{f(x)}{g(y)} \overset{\text{separate}}{\rightsquigarrow} g(y)\,dy = f(x)\,dx$$

$$\Big\downarrow \text{ integrate}$$

$$\int g(y)\,dy = \int f(x)\,dx$$

In integration, different methods (u-sub, by parts).

3. Optimization basics

- Local/relative extrema : find critical points

1 var : critical pt at $x=a$ if either $f'(a)=0$ or $f'(a)$ DNE , but a in domain.

2 var : crit pt at $(x,y)=(a,b)$ if both $f_x(a,b)=0$ and $f_y(a,b)=0$.

- Classifying critical points

in 1 variable : 1^{st} derivative test
2^{nd} derivative test

in 2 vars : 2^{nd} partial deriv. test
$(D(x,y)=f_{xx}\cdot f_{yy}-(f_{xy})^2)$

4. Antiderivatives : Only in 1 variable.

$F(x)$ antideriv. of $f(x)$ if $F'(x) = f(x)$

$$F(x) \xrightarrow{\text{deriv.}} f(x) = F'(x)$$

integration

$$\int f(x)\,dx = F(x) + \underline{\ C\ }$$

Concepts to review:

1. Integration techniques : u-sub, by parts.
2. Definite integrals

③ initial value problems.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(no constant!)

Caution: if using u-sub, change bounds/limits of integration

$$x \longrightarrow u(x)$$
$$b \longrightarrow u(b)$$
$$a \longrightarrow u(a)$$

3. Properties of definite integrals

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

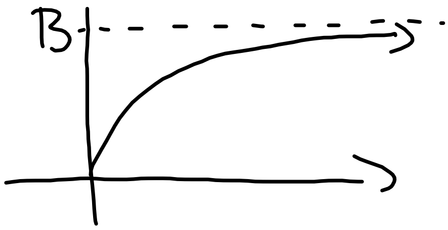
Application Topics

1. Exponential models

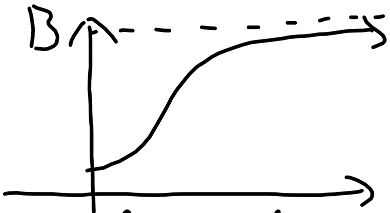
Concepts to review:

1. Learning and logistic curves

Learning curve : $B - Ae^{-kt}$



Logistic curve : $\dfrac{B}{1+Ae^{-Bkt}}$



"Least upper bound of max" = value of asymptote.

look at limit as $t \to \infty$

2. Half-life / radioactive decay

$$Q(t) = Q_0 e^{-kt}$$

If you have the half life: time t_0

for which $\frac{1}{2}\cancel{Q_0} = \cancel{Q_0} e^{-kt_0}$ solve for k

2. Present and future values (continuous compounding)

① Simplest case: deposit principal P_0 once, at the beginning.

future value $B(t) = FV(t) = P_0 \cdot e^{rt}$

② Income stream $f(t)$, after T years,

$$FV(T) = e^{rT} \int_0^T f(t) e^{-rt}\, dt$$

$$PV(T) = \frac{FV(T)}{e^{rT}} = \int_0^T f(t) e^{-rt}\, dt$$

3. Marginal analysis

Used for estimating values of functions if we slightly vary the input

- In 1 variable: if we change input by 1, and x is large enough, $f(x+1) - f(x) \approx f'(x)$

Generally, if we change x by Δx,

$$f(x+\Delta x) - f(x) = \Delta y \approx f'(x)\,\Delta x$$

- In 2 variables: $z = f(x, y), \Delta z =$ change in z, then

$$f(x+1, y) - f(x \cancel{+1}, y) \approx f_x(x, y)$$
$$f(x, y+1) - f(x, y \cancel{+1}) \approx f_y(x, y)$$

General formula : $$\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

4. Optimization (absolute largest / smallest value)

Used for maximizing profit, output, minimizing costs etc.

- In 1 variable: • on a closed interval, use Extreme value thm.
  - in general, sketch the graph.
- In 2 variables:
  - on closed bounded region, use Extreme value property in 2 variables
  - if subject to constraint $g(x,y)=k$, use Lagrange multipliers.

5. Interpreting the definite integral

Idea: approximating an accumulation of objects by summing over small intervals is the same as calculating a definite integral. The main example is calculating an area.

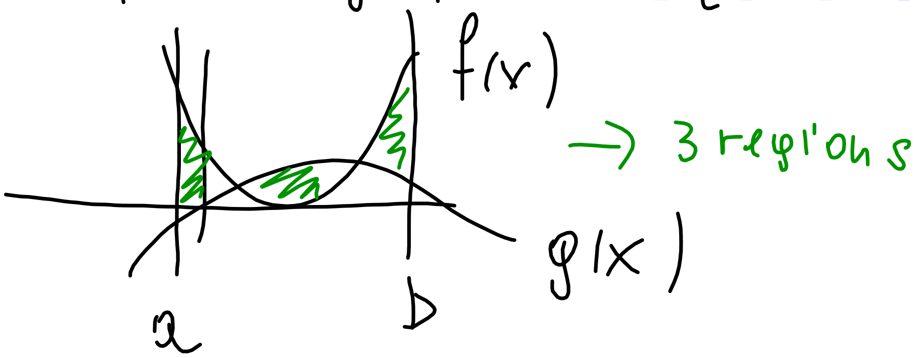
Concepts to review:

1. Areas : ① b/w $f(x)$ and x-axis

   if $f(x) \geq 0$, area is $A = \int_a^b f(x)\,dx$

② b/w graphs of $f(x)$ and $g(x)$, and maybe $x = a$, $x = b$

→ divide into regions where you know whether $f(x)$ or $g(x)$ is above



5. Interpreting the definite integral (cont.)

2. Average values